Unified Covariate Adjustment:

Estimating Multilinear Causal Estimands

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School of Information Science (iSchool)

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Motivation: Multilinear Causal Estimands

A causal effect $\mathbb{E}[Y \mid \mathrm{do}(\overline{\mathbf{a}})]$ is often identified as a multilinear functional.

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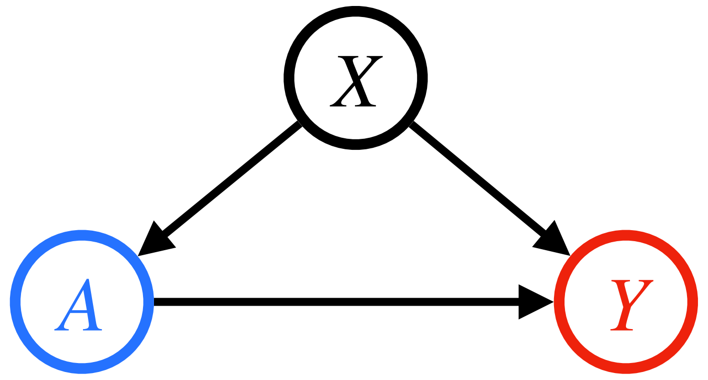

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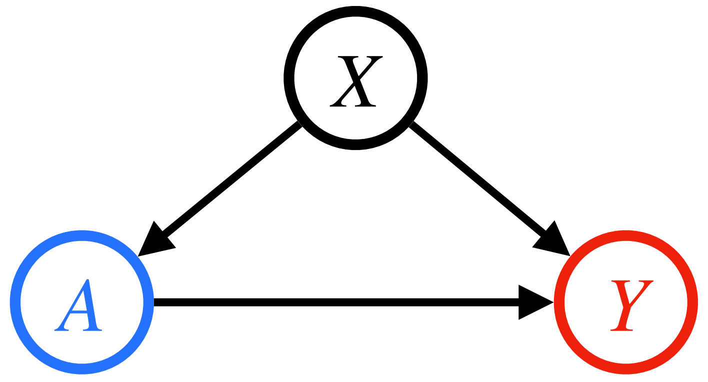

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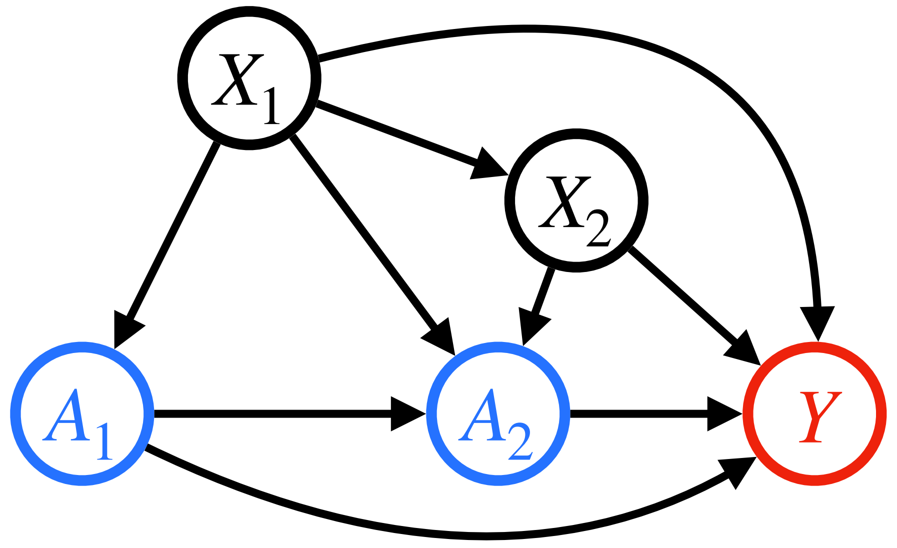

$$\sum_{x_1,x_2} \mathbb{E}[Y \mid \bar{a}_2, x_2, \bar{a}_1, x_1]P(x_2 \mid \bar{a}_1, x_1)P(x_1)$$

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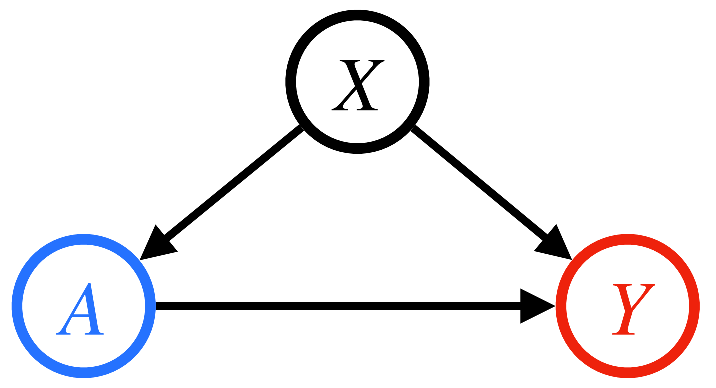

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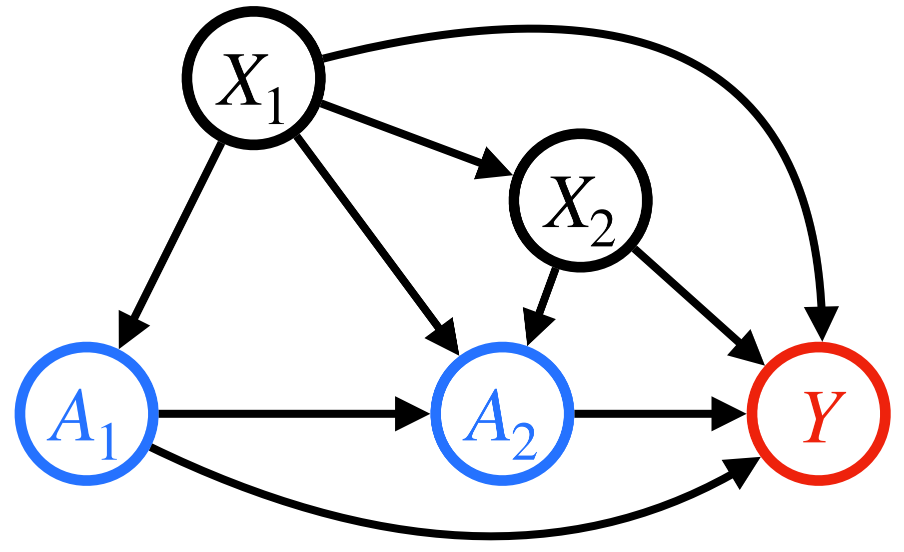

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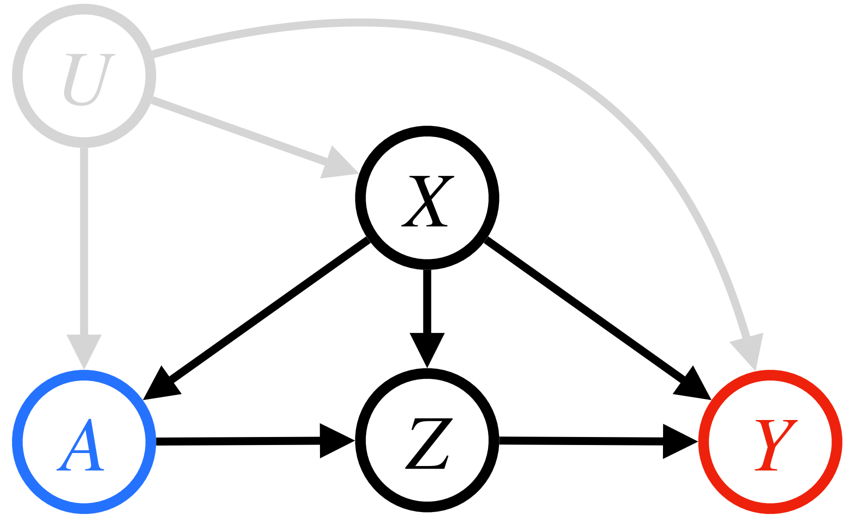

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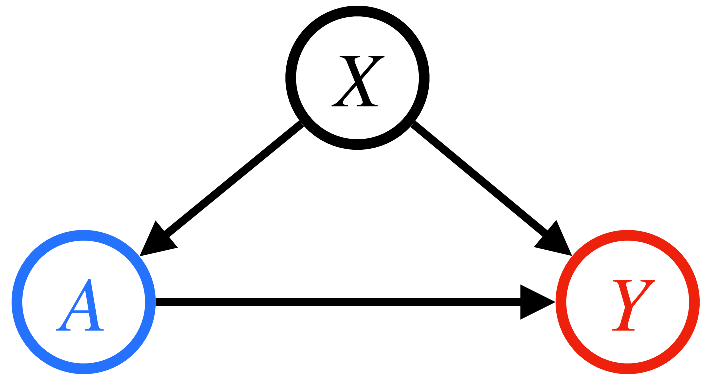

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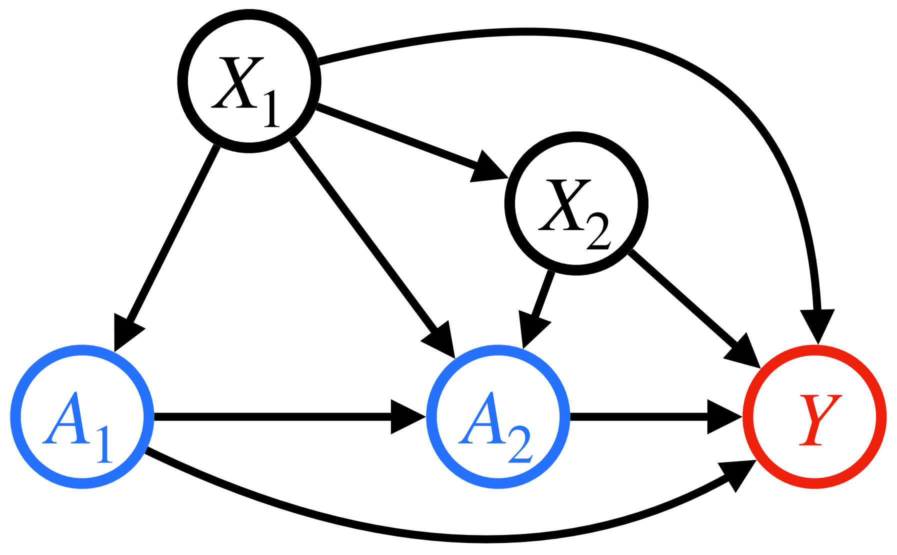

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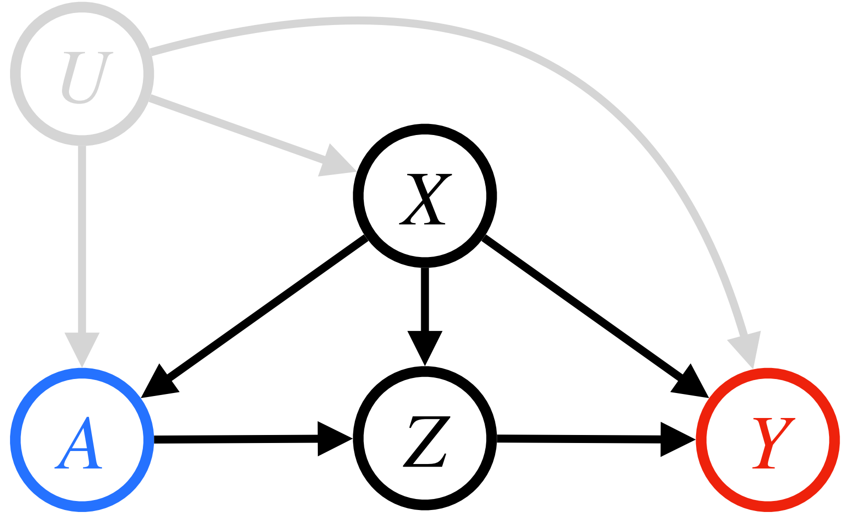

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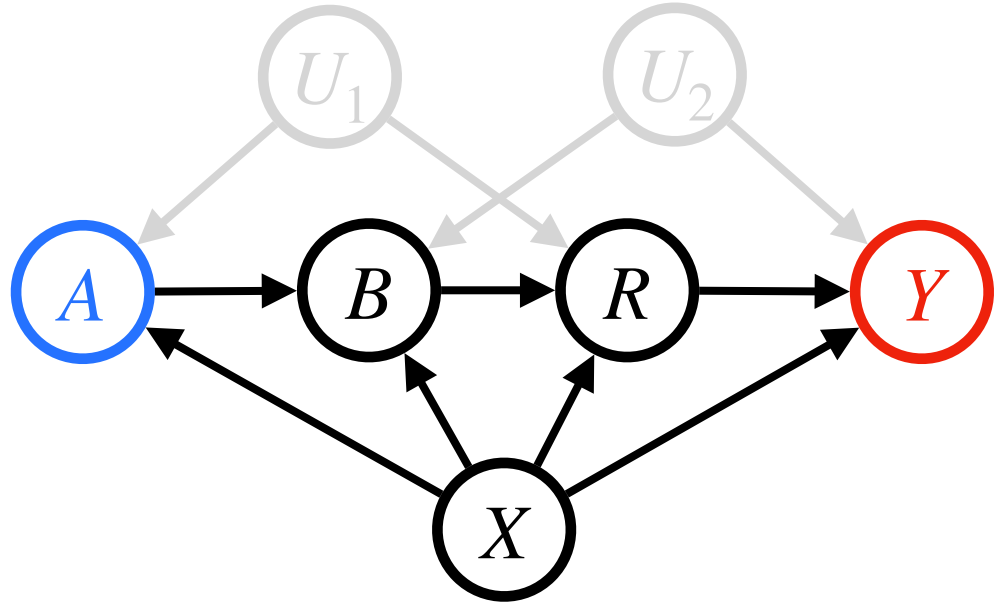

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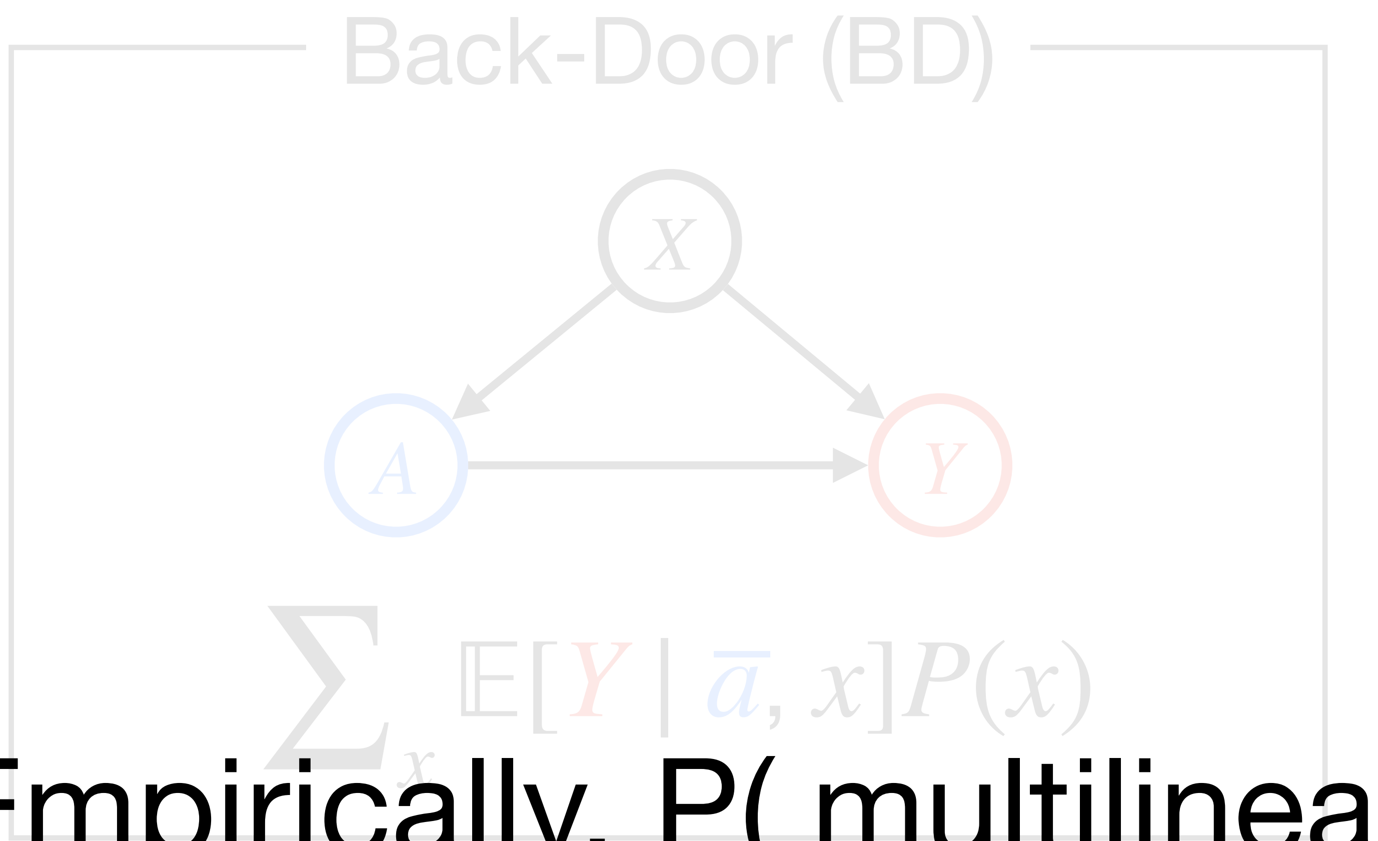
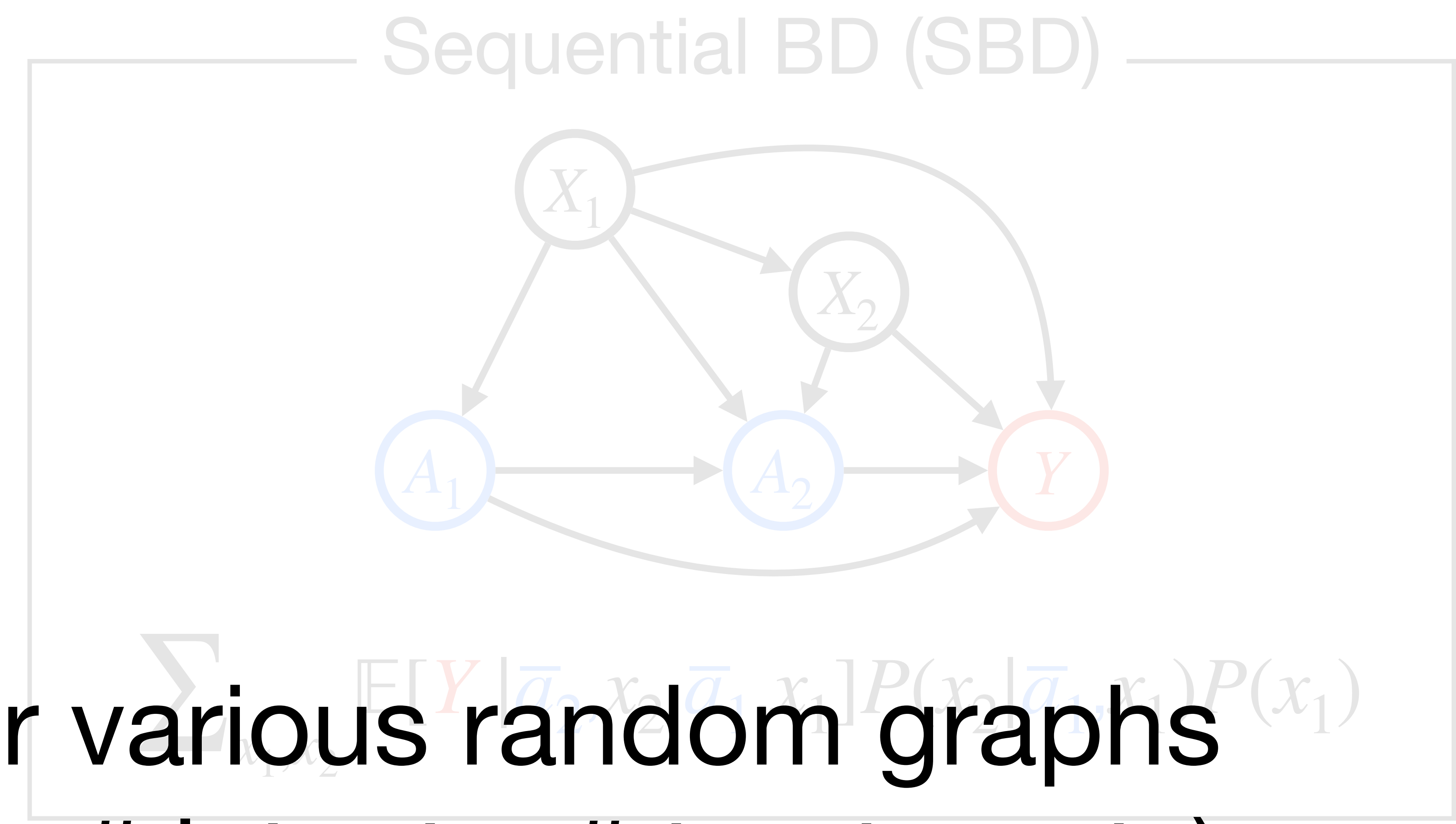
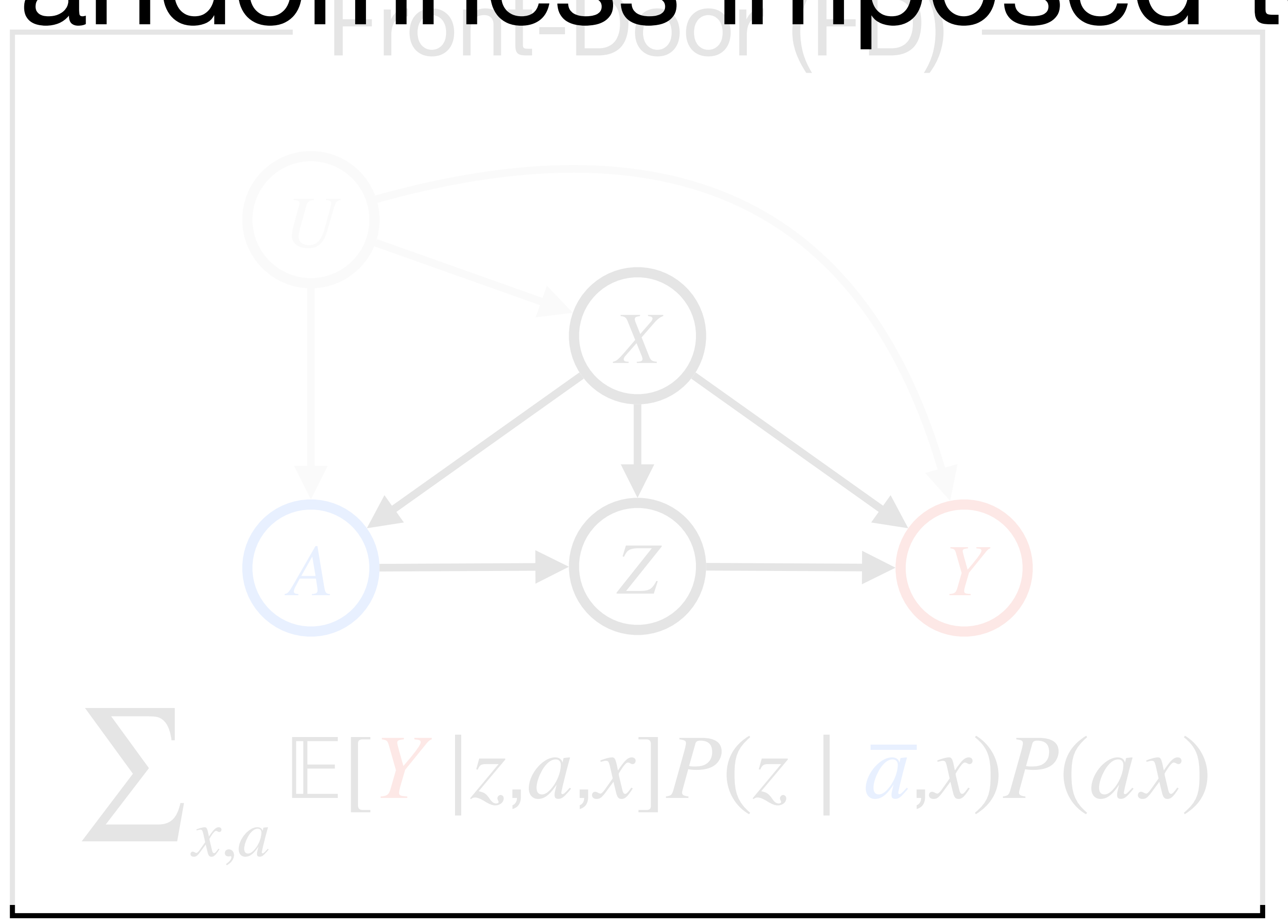
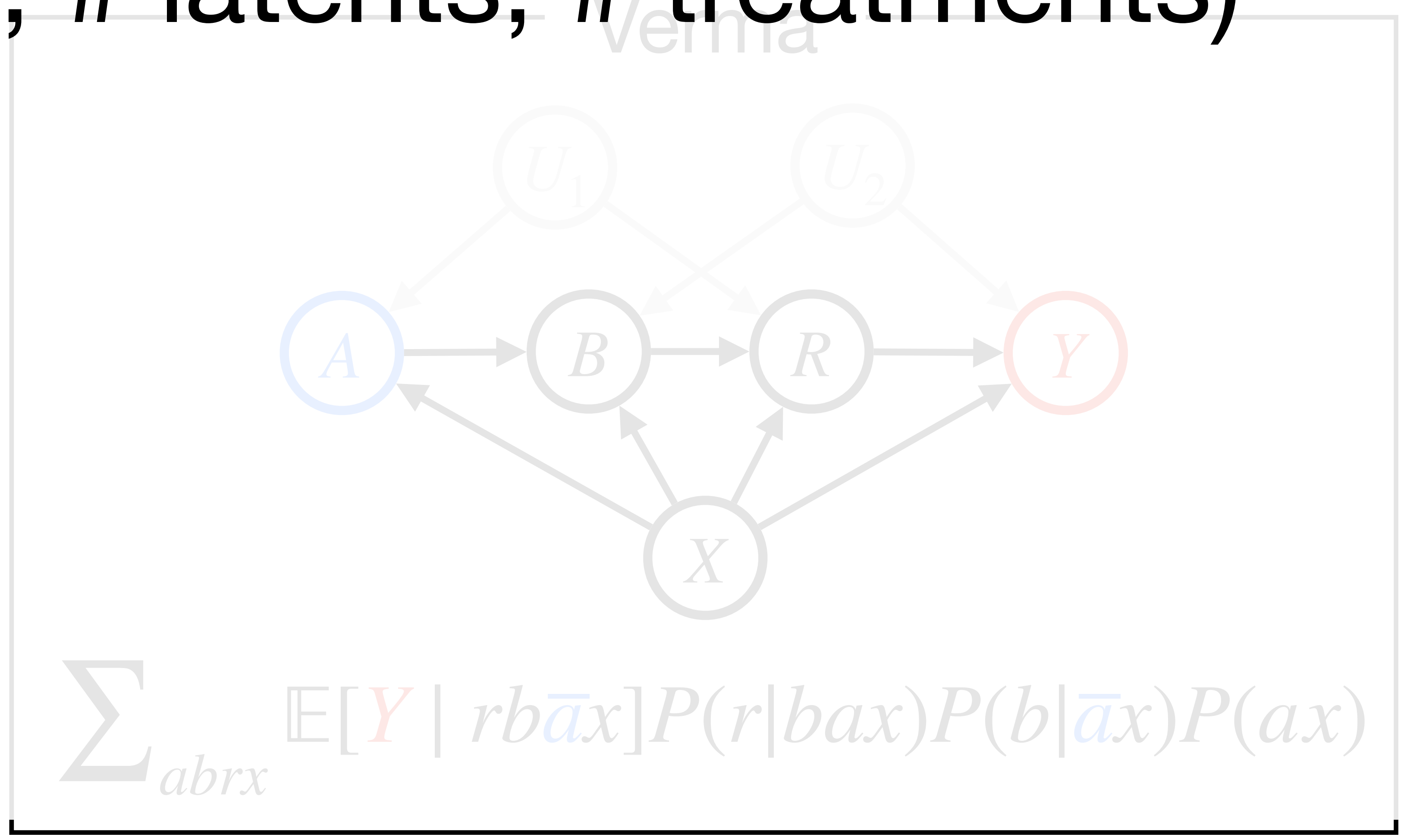




Empirically, P(multilinear | ID) > 99% for various random graphs
(randomness imposed to # observables, # latents, # treatments)





Tasks

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1. **Identification**: Sound and complete graphical criterion for identifying causal effects as a multilinear causal estimand

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❸ **Sample efficiency:** A doubly robust and sample efficient estimation framework

Identification Criterion for Multilniear Estimands

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Implication of Multilinear Causal Estimands Criterion

Suppose a causal effect $\mathbb{E}[Y \mid \mathrm{do}(\overline{\mathbf{a}})]$ is identifiable. Its ID expression is multilinear, iff, it is given as a product of SBDs.

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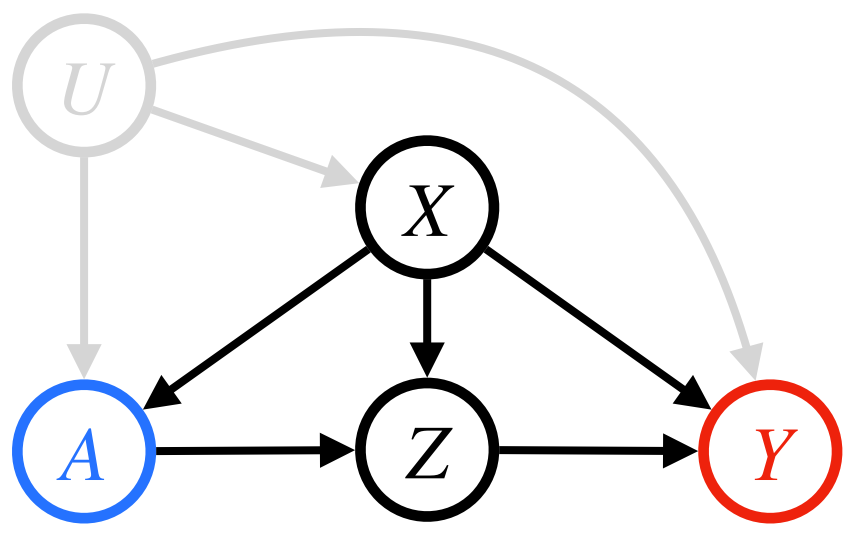

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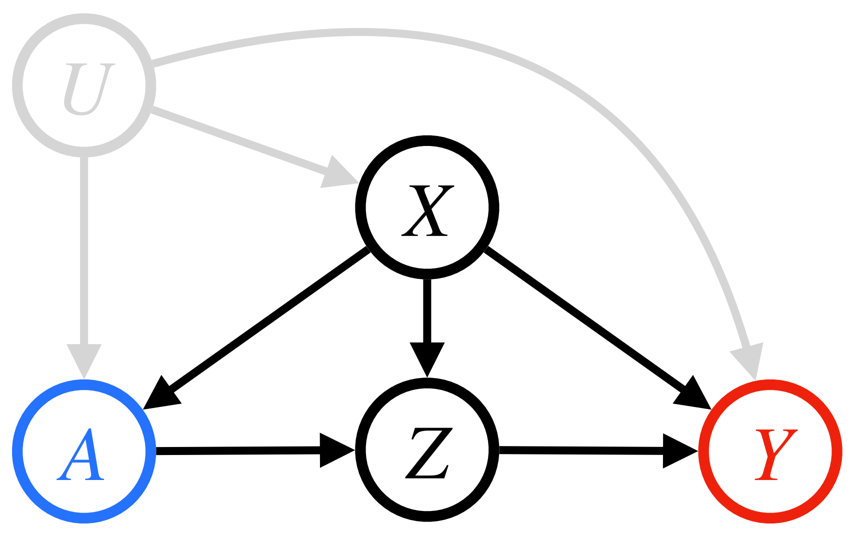

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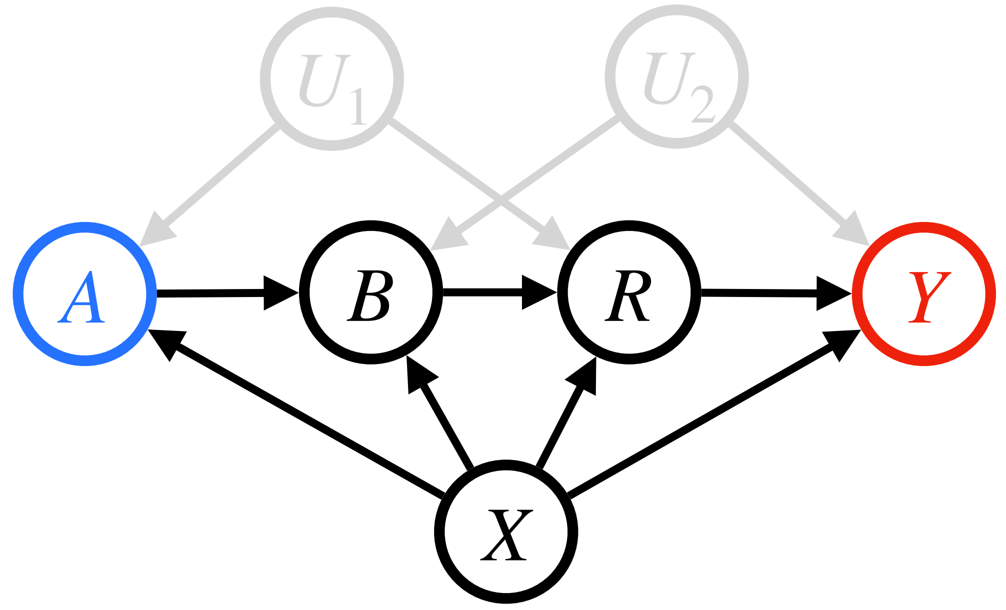

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Nested Conditional Expectation for SBD

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Evaluating $\sum_{x_1, x_2}$ is computationally expensive, but can be circumvented by nested expectation.

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The SBD estimand can be estimated in a computationally efficient manner using nested conditional expectations.

Limitation of Nested Expectation: FD

Front-Door (FD): $\mathbb{E}[Y \mid \mathrm{do}(\overline{a})] \triangleq \sum_{x,a} \mathbb{E}[Y \mid z,a,x] P(z \mid \overline{a},x) P(x)$

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The standard nested expectation cannot represent multilinear estimands when treatments are both marginalized and fixed simultaneously.

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Treatments $\mathbf{A}$ are fixed to $\bar{\mathbf{a}}$ and marginalized $\mathbf{a}$ simultaneously.

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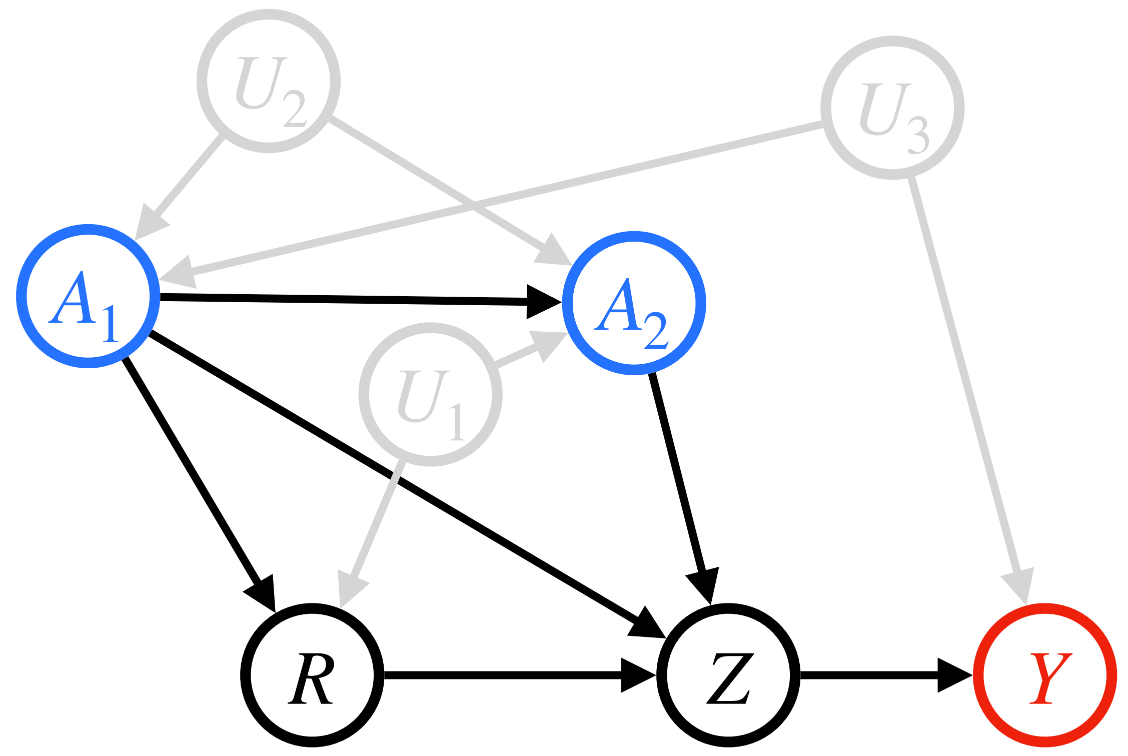
Example



$\sum_{r,z,a_1,a_2,r'} \mathbb{E}[Y \mid z, r, a_2, a_1] P(z \mid r,\bar{a}_1,\bar{a}_2) P(r \mid \bar{a}_1) P(r,a_1,a_2)$

$\exists$variable that is marginalized multiple times.

Kernel Policy Product: Representation of MCE

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Kernel Policy Product: A product of conditional distributions and policies over variables & their copied proxies

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Computational efficiency gain via replacing $\sum_{z,a,x}$ through KPP expression with copied proxies and nested conditional expectations

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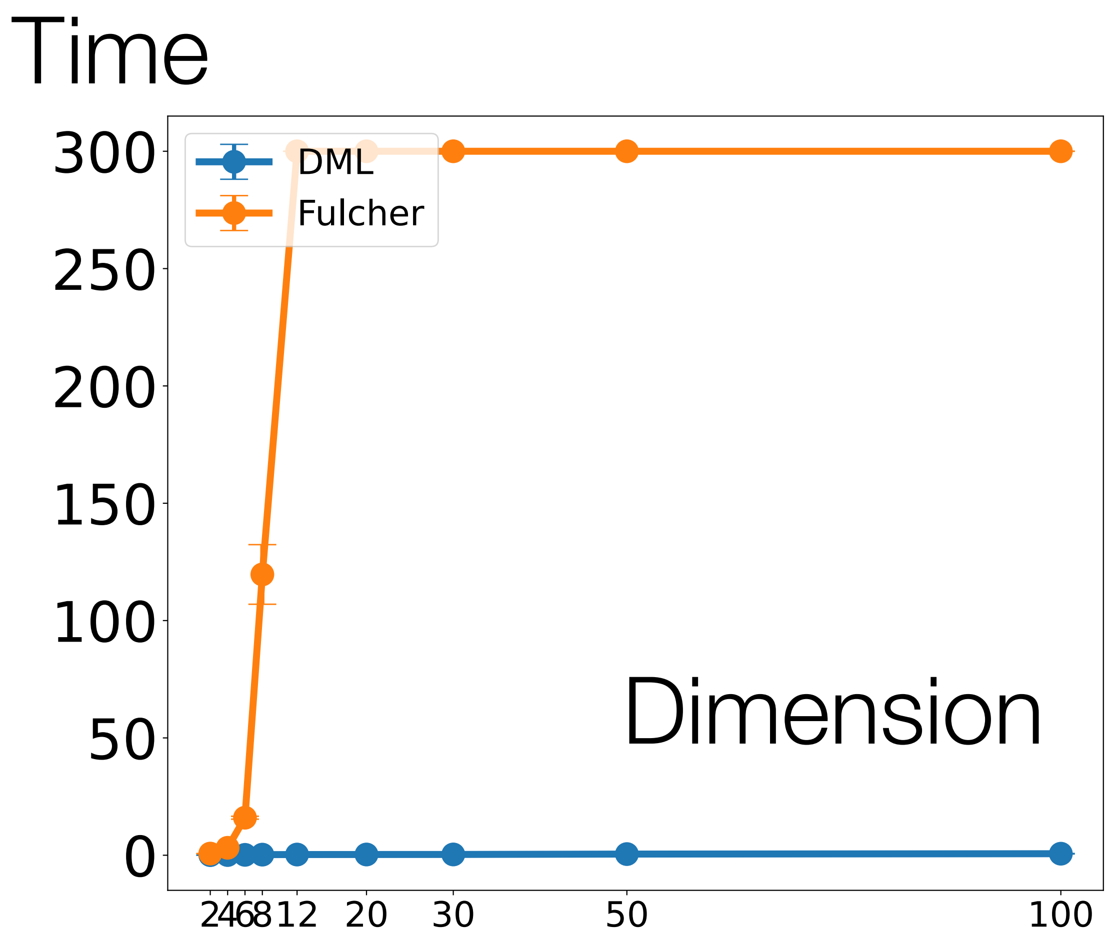
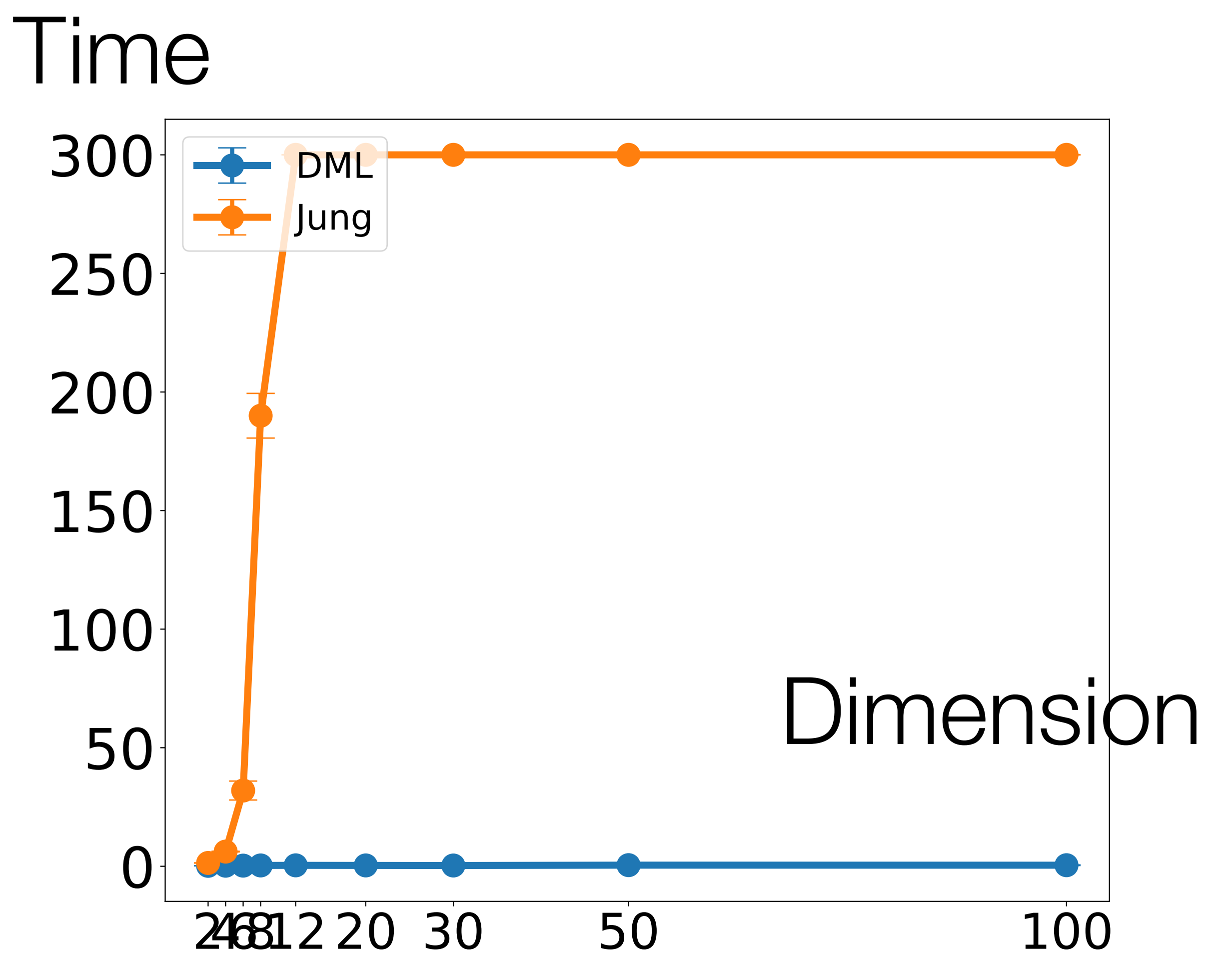
2. UCA($\hat{\mu}$,$\hat{\pi}$) is an AIPW-style estimator tailored to KPP.

$$\mathrm{Error}(\mathrm{UCA}(\hat{\boldsymbol{\mu}},\hat{\boldsymbol{\pi}}), \mathrm{DML}(\boldsymbol{\mu},\boldsymbol{\pi})) = \sum_i \mathrm{Error}(\hat{\mu}_i,\mu_i) \times \mathrm{Error}(\hat{\pi}_i,\pi_i)$$

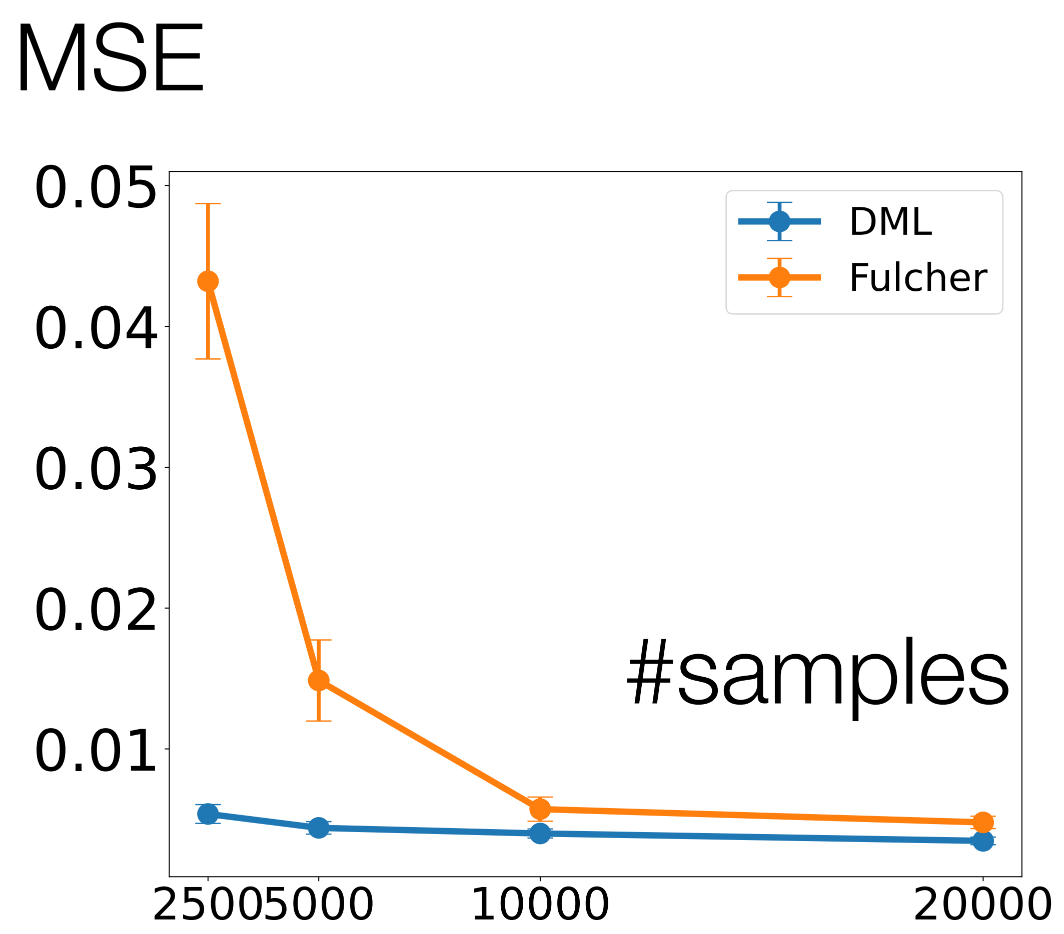
Simulation Results

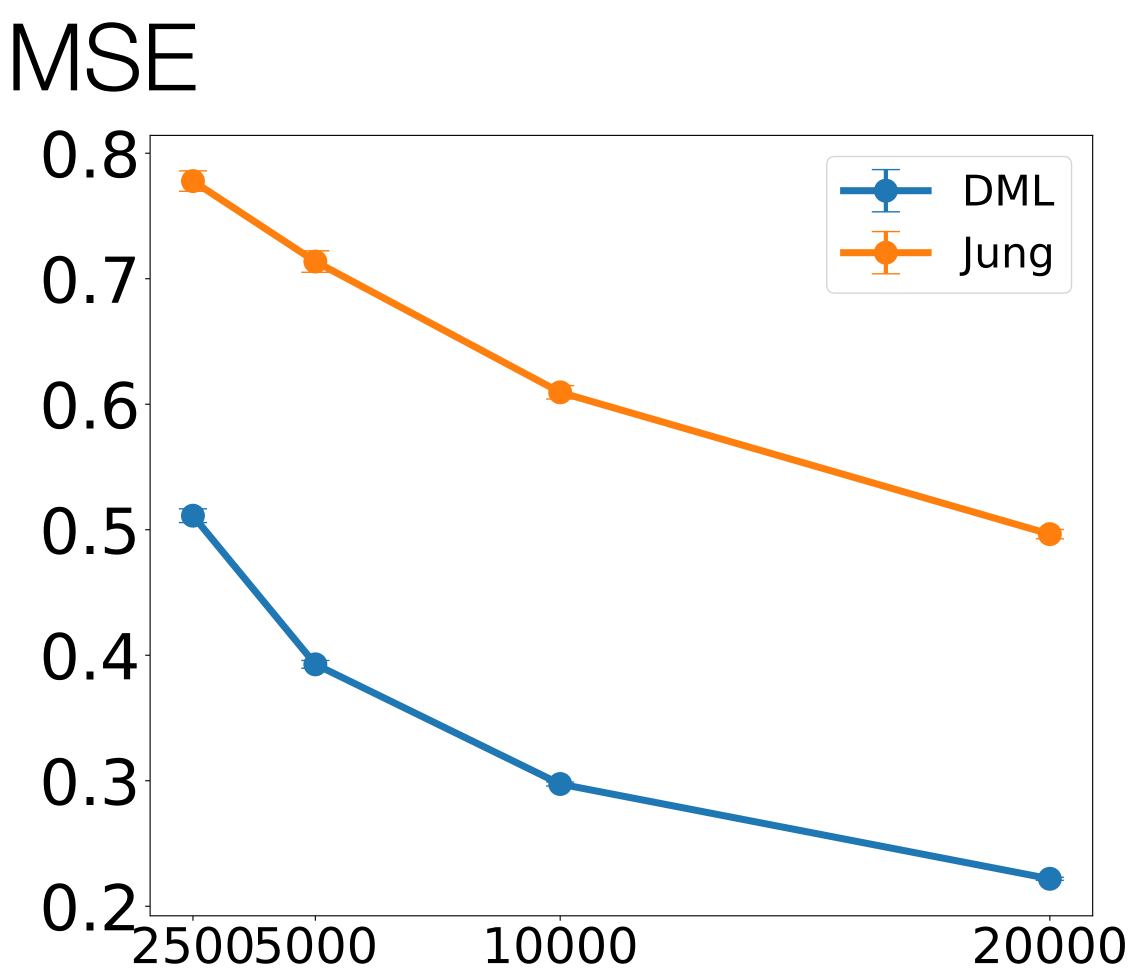
FD | **Verma**





- Existing estimators' evaluation time increase as dimensions increases
- UCA estimator exhibits computational efficiency gains.





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❷ **Computationally efficiency**: A new formulation for multilinear causal estimands with computational efficiency.

❸ **Sample efficiency:** A doubly robust and sample efficient estimation framework