Estimating Causal Effects

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Introduction

Outline

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- **Part 1**: Double/Debiased Machine Learning (DML)

  Chernozhukov, Victor, et al. "Double/debiased machine learning for treatment and structural parameters: Double/debiased machine learning." The Econometrics Journal 21.1 (2018).

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- **Part 1**: Double/Debiased Machine Learning (DML)

  Chernozhukov, Victor, et al. "Double/debiased machine learning for treatment and structural parameters: Double/debiased machine learning." The Econometrics Journal 21.1 (2018).

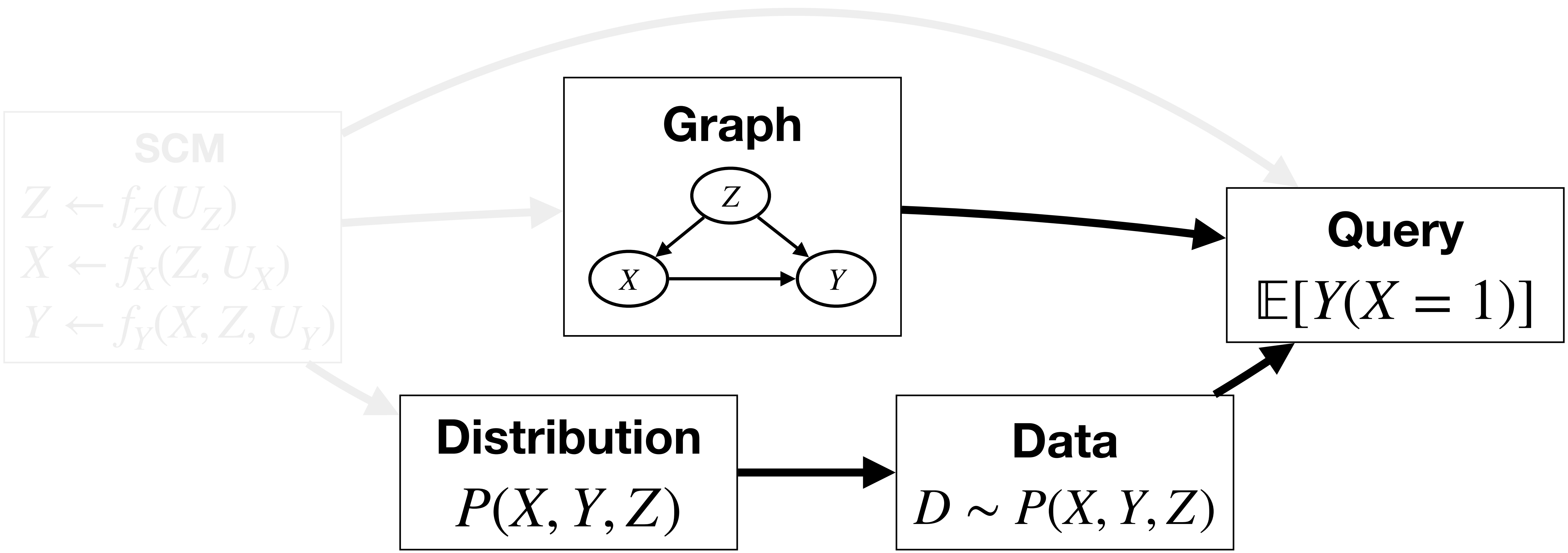
- **Part 2**: DML for any identifiable functional

  Jung, Tian, Bareinboim (2021a). Estimating Identifiable Causal Effects through Double Machine Learning. In Proceedings of the 35th AAAI Conference on AI, 2021.

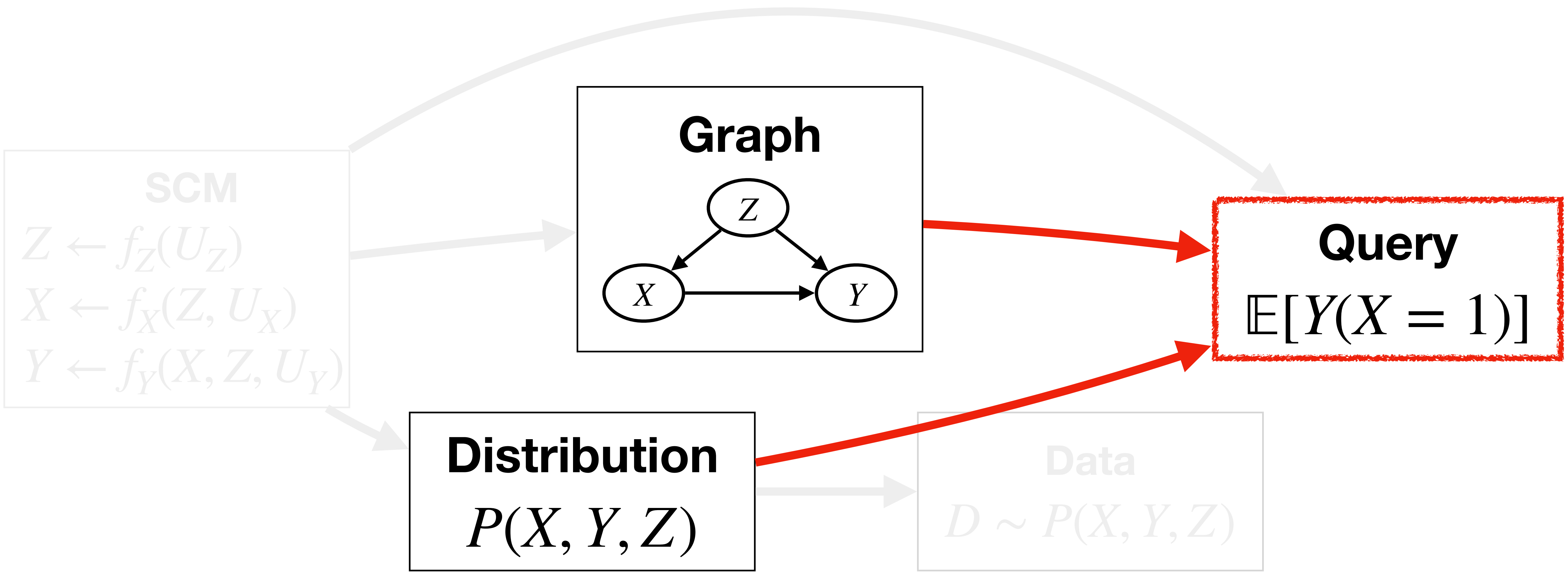
Part I.
Double/Debiased Machine Learning

Big Picture In Causal Inference

Big Picture for Causal Inference: Inaccessibility to SCMs



Causal Effect Identification: Big Picture (1)



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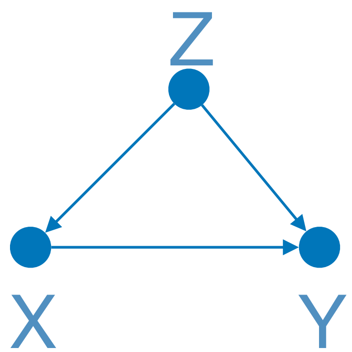


Causal graph (G)

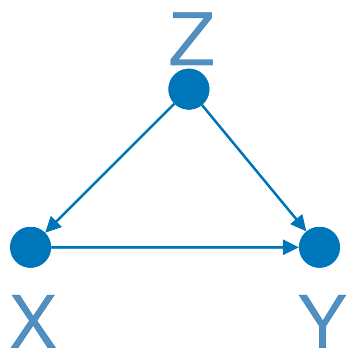
$$P(Z, X, Y)$$

Distribution on G (P)

$$Q_0 := \mathbb{E}[Y \mid do(x)]$$

Causal Query (Q_0)

Causal Effect Identification



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Given {G, P, Q_0},

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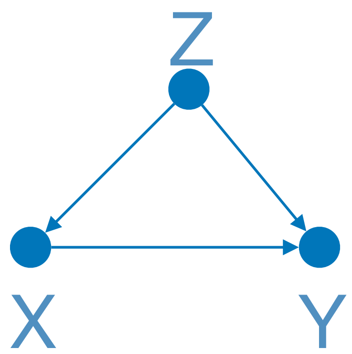

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Given {G, P, Q_0},

the causal effect identification (ID) task

$$\mathbf{ID}(G, P, Q_0)$$

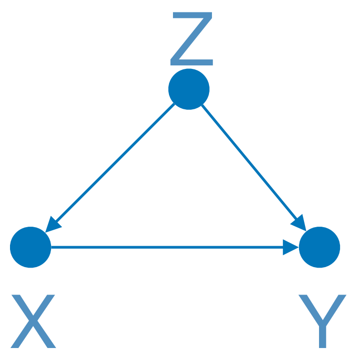
ID algorithm

Causal Effect Identification

Given {G, P, Q_0},

the causal effect identification (ID) task

finds the functional $C(P)$ s.t. $C(P) = Q_0$.



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Causal Query (Q_0)

ID(G, P, Q_0)

ID algorithm

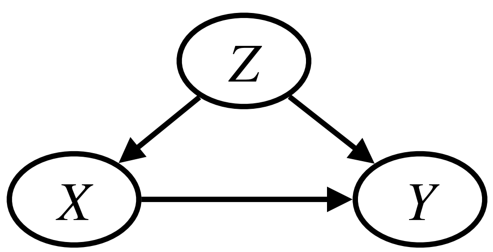
$C(P) = \sum_z \mathbb{E}[Y \mid x, z] P(z)$

Causal functional

$C(P)$ **s.t.** $C(P) = Q_0$

Causal Effect Identification: Definition

Graph (G)



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(e.g., $P(Z, X, Y)$)

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(e.g., $\mathbb{E}[Y(x)]$)

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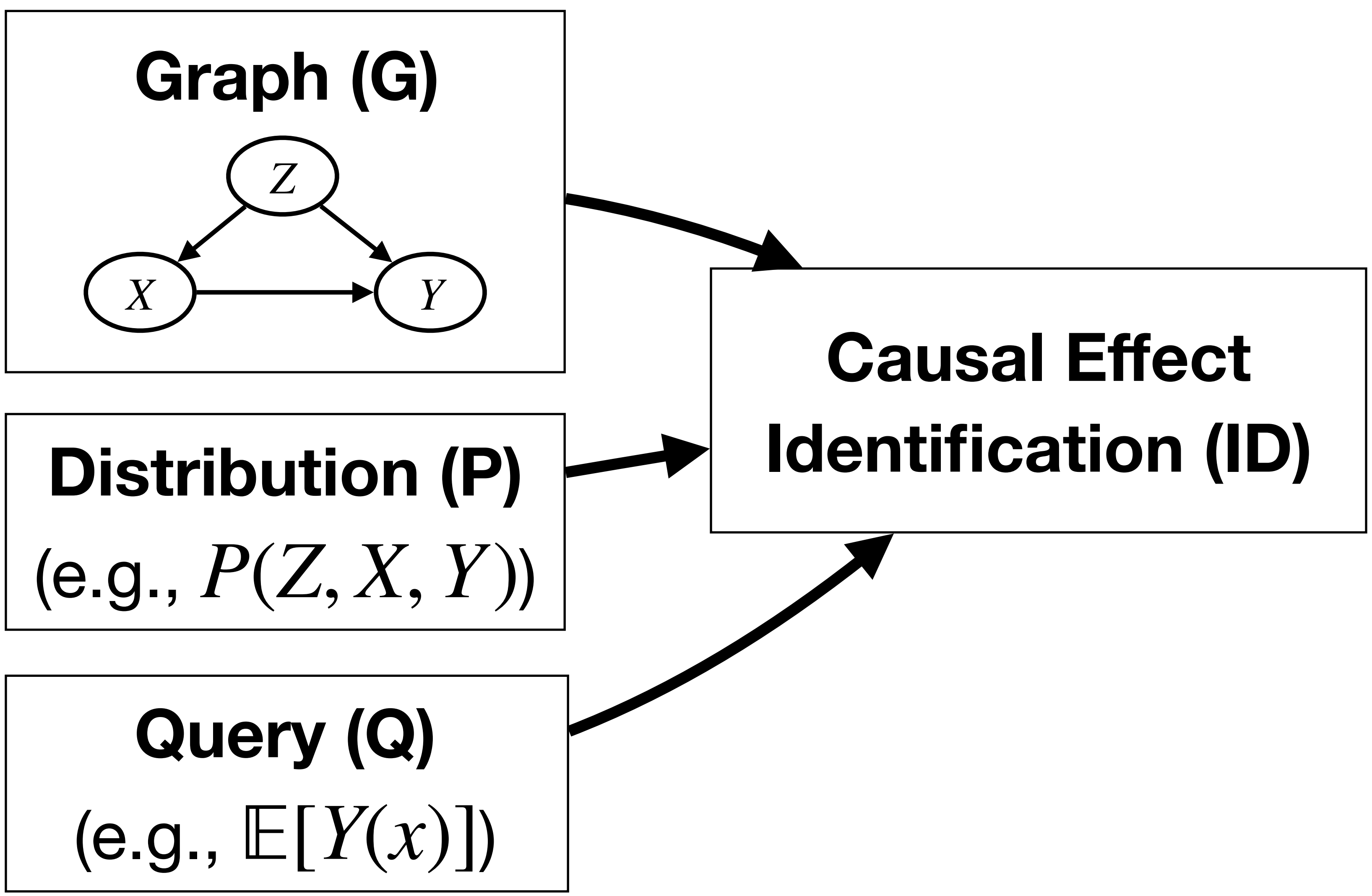

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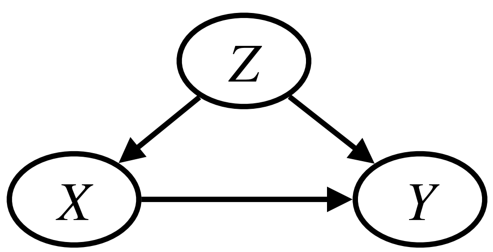

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(e.g., $\mathbb{E}[Y(x)]$)

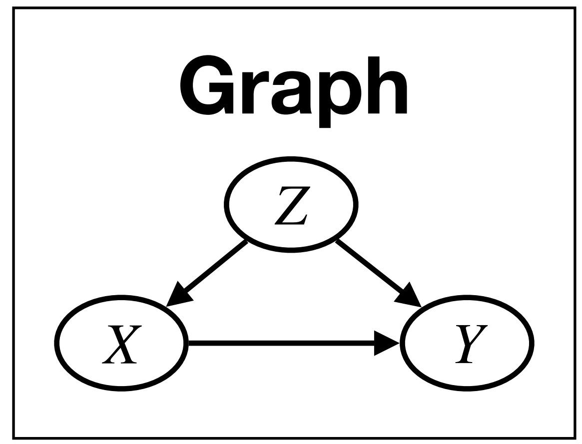
Causal Effect Identification (ID)

Solution

YES: A function $C(P)$ s.t. $C(P) = Q$.

No: If no such functions exist.

Causal Effect Estimation: Back-door Adjustment



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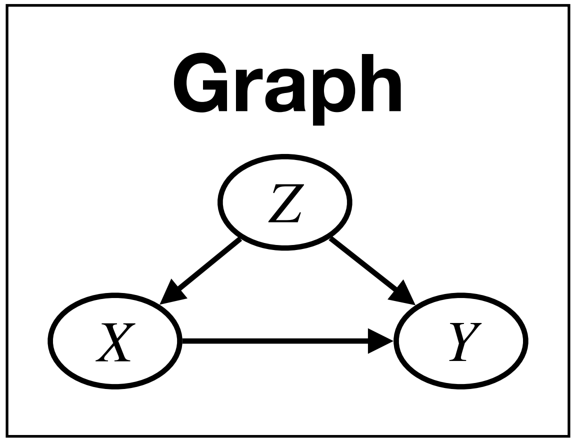

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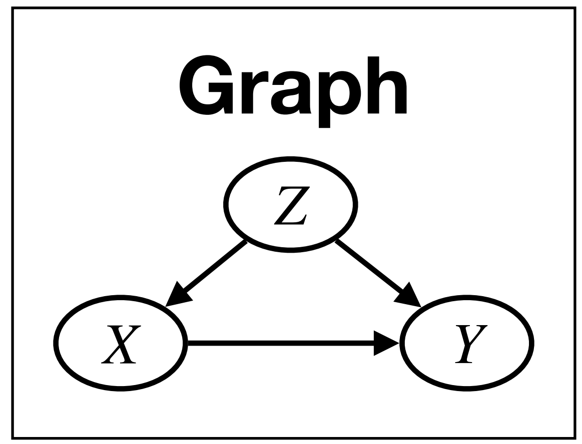

Back-door Adjustment

↳ If there exists Z s.t. (1) Z is non-descendent of $\{X, Y\}$ and (2) $(Y \perp\!\!\!\perp X \,|\, Z)_{\underline{X}}$, then

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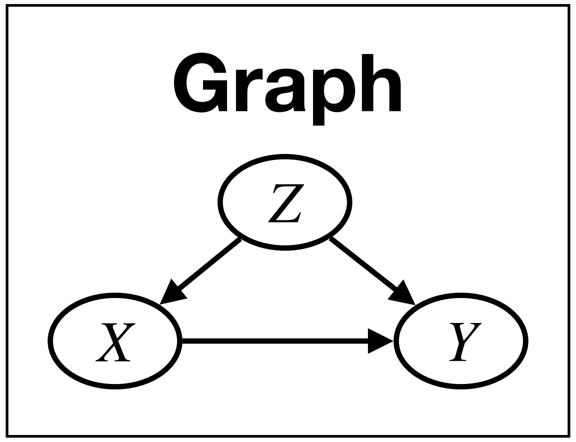

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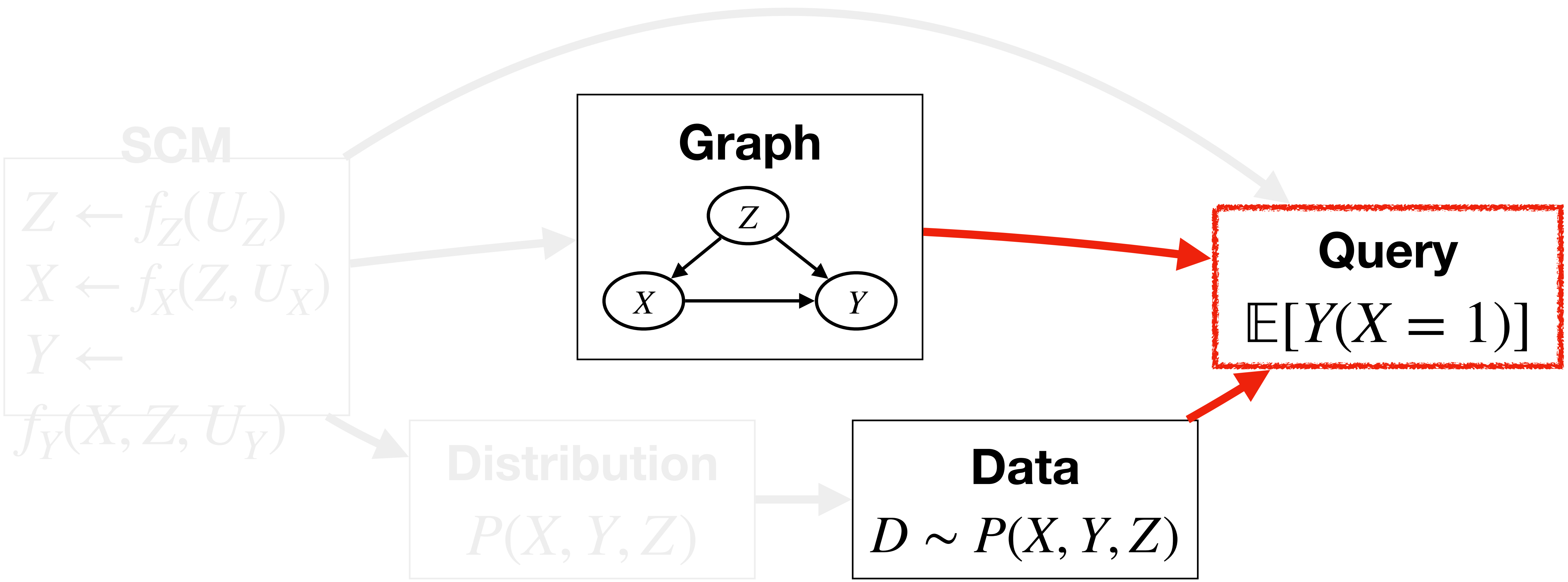
↳ If there exists Z s.t. (1) Z is non-descendent of $\{X, Y\}$ and (2) $(Y \perp\!\!\!\perp X \mid Z)_{\underline{X}}$, then

$$\mathbb{E}[Y(x)] = \sum_z \mathbb{E}[Y \mid x, z] P(z).$$

Causal Effect Estimation: Big Picture (1)



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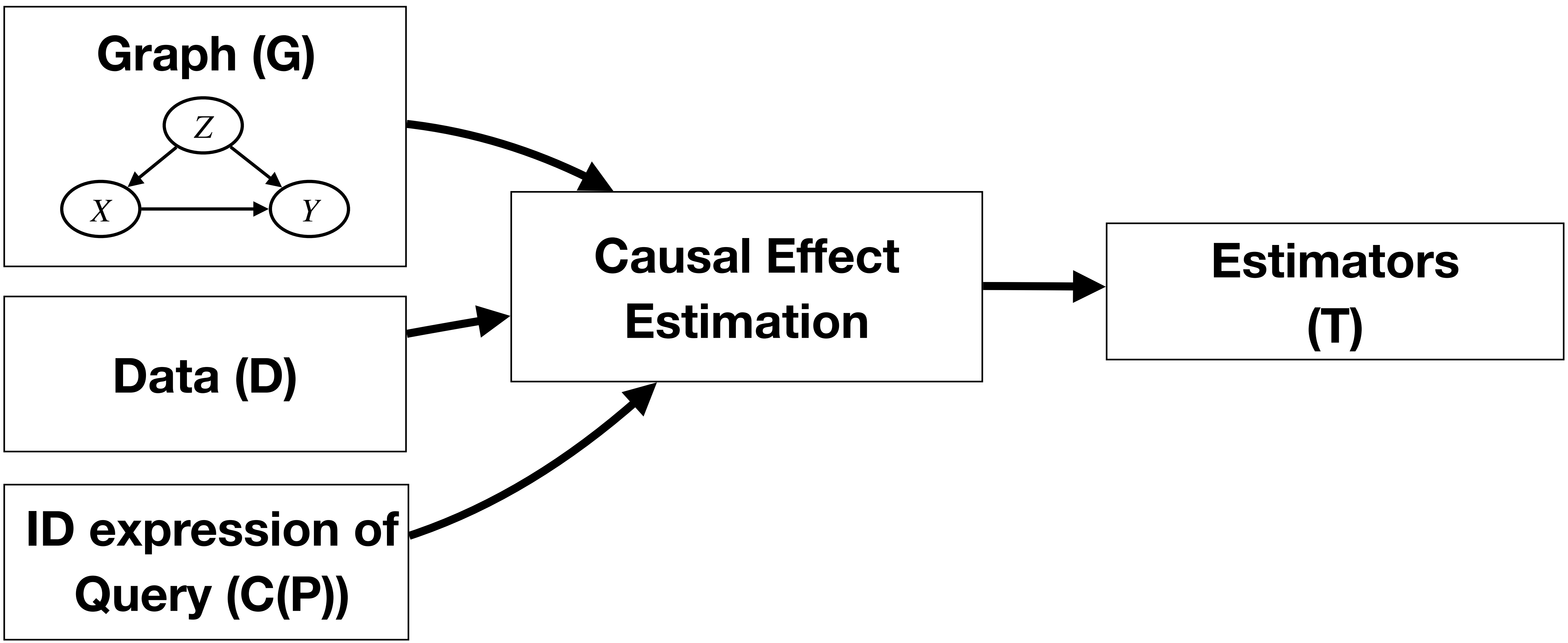
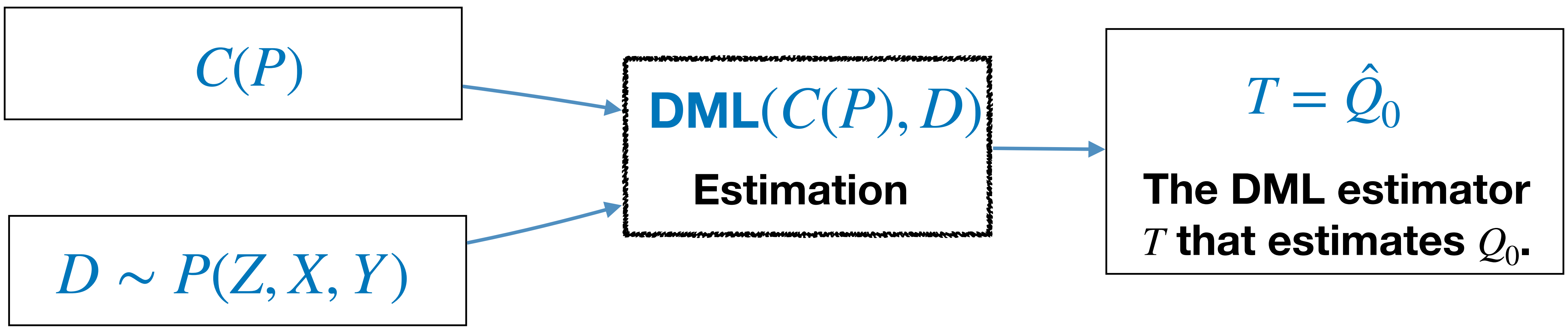


Toward Double/Debiased Machine Learning

Double/Debiased Machine Learning (DML) [Chernozhukov et al., 2018] is a framework of constructing the estimator T.



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We will understand the mechanism of the DML estimator with the BD adjustment example:

$$C(P) = \sum_z \mathbb{E}[Y | x, z] P(z) = \mathbb{E}[Y | do(x)]$$

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We assume the followings in the lecture.

$X \in \{0,1\}$ a binary treatment variable.

$P(v) > 0$ for any v.

Z can be multivariate (continuous/discrete)

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$= P(y \mid do(x))$ when $Y \leftarrow I_y(Y)$, an indicator function that is 1 when $Y = y$

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Estimand: Amenable Expression for Estimation

Challenges in estimating $C(P)$

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$$C(P) = \sum_z \mathbb{E}[Y \,|\, x, z] P(z)$$

Estimating $C(P)$ directly is challenging when Z is

- high-dimensional or
- a mixture of continuous/discrete variables.

Causal Estimand

$$C(P) = \sum_z \mathbb{E}[Y \mid x, z] P(z)$$

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Estimand $f(V; \eta)$

An arbitrary function of V, η s.t. $\mathbb{E}[f(V; \eta_0)] = C(P)$ when $\eta = \eta_0$ for some η_0.

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Next, we will introduce three estimands for the BD adjustment case.

Regression (REG)-based Estimand

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Let $\mu(X, Z)$ be any arbitrary function of $\{X, Z\}$, and $\mu_0(X, Z) := \mathbb{E}[Y \mid X, Z]$.

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Then, with the true nuisance μ_0,

$$\mathbb{E}[f^{REG}(V; \mu_0)] = \mathbb{E}[\mu_0(x, Z)] = \mathbb{E}[\mathbb{E}[Y \mid x, Z]] = C(P)$$

Inverse Probability Weighting (IPW)-based Estimand - 1

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Indicator s.t. 1 when x' is equal to x

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$$= \mathbb{E}_{X,Y,Z} \left[\frac{I_x(X)}{P(X \mid Z)} Y \right] \quad \text{(Shortly, } = \mathbb{E} \left[\frac{I_x(X)}{P(X \mid Z)} Y \right] \text{)}$$

Inverse Probability Weighting (IPW)-based Estimand - 2

$$C(P) = \mathbb{E}\left[\frac{I_x(X)}{P(X \mid Z)} Y\right]$$

from the prev. slide.

$$C(P) = \sum_z \mathbb{E}[Y \mid x, z] P(z)$$

$$\mathbb{E}[f(V; \eta_0)] = C(P)$$

Inverse Probability Weighting (IPW)-based Estimand - 2

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Let $\pi(X|Z)$ be an arbitrary positive function and $\pi_0(X|Z) := P(X|Z)$.

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Then, with the true nuisance π_0,

$$\mathbb{E}[f^{IPW}(V;\pi_0)] = \mathbb{E}\left[\frac{I_x(X)}{\pi_0(X|Z)}Y\right] = \mathbb{E}\left[\frac{I_x(X)}{P(X|Z)}Y\right] = C(P)$$

Toward a
Doubly Robust Estimand

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Now, we will introduce a "*doubly robust (DR)*" estimand, denoted

$$f^{DR}(V; \{\pi, \mu\})$$

where the name is from a property known as "*doubly robustness*":

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Doubly robustness implies we have double chance for being correct in terms of the estimation of the nuisances parameters.

Derivation of Doubly Robust Estimand - 1

$$C(P) = \sum_z \mathbb{E}[Y \mid x, z] P(z)$$

$$\mathbb{E}[f(V; \eta_0)] = C(P)$$

$$f^{REG}(V; \eta_0 := \mu) := \mu(x, Z)$$

$$f^{IPW}(V; \eta := \pi) := \frac{I_x(X)}{\pi(X \mid Z)} Y$$

Derivation of Doubly Robust Estimand - 1

$$C(P) = \mathbb{E}\left[f^{IPW}(V; \pi_0)\right]$$

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Derivation of Doubly Robust Estimand - 1

$$C(P) = \mathbb{E}\left[f^{IPW}(V;\pi_0)\right]$$

$$+\mathbb{E}[f^{REG}(V;\mu_0)]$$

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$$+\mathbb{E}[f^{REG}(V;\mu_0)]$$ = C(P)

$$-\mathbb{E}\left[\frac{I_x(X)}{\pi_0(X|Z)}\mu_0(X,Z)\right]$$

$$C(P) = \sum_z \mathbb{E}[Y|x,z]P(z)$$

$$\mathbb{E}[f(V;\eta_0)] = C(P)$$

$$f^{REG}(V;\eta_0 := \mu) := \mu(x,Z)$$

$$f^{IPW}(V;\eta := \pi) := \frac{I_x(X)}{\pi(X|Z)}Y$$

Derivation of Doubly Robust Estimand - 1

$$C(P) = \mathbb{E}\left[f^{IPW}(V;\pi_0)\right]$$ = C(P)

$$+\mathbb{E}[f^{REG}(V;\mu_0)]$$ = C(P)

$$-\mathbb{E}\left[\frac{I_x(X)}{\pi_0(X|Z)}\mu_0(X,Z)\right]$$

= C(P), as shown next

$$C(P) = \sum_z \mathbb{E}[Y|x,z]P(z)$$

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Derivation of Doubly Robust Estimand - 2

$$C(P) = \sum_z \mathbb{E}[Y \mid x, z] P(z)$$

Derivation of Doubly Robust Estimand - 2

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$$\mathbb{E}\left[\frac{I_x(X)}{\pi_0(X \mid Z)}\mu_0(X, Z)\right] = \sum_{x', z} \frac{I_x(x')}{\pi_0(x' \mid z)} \underbrace{\mu_0(x', z)}_{=\mathbb{E}[Y|x,z]} \underbrace{P(x' \mid z)}_{=\pi_0(x'|z)} P(z)$$

Derivation of Doubly Robust Estimand - 2

$$C(P) = \sum_z \mathbb{E}[Y \,|\, x, z] P(z)$$

$$\mathbb{E}\left[\frac{I_x(X)}{\pi_0(X \,|\, Z)} \mu_0(X, Z)\right] = \sum_{x',z} \frac{I_x(x')}{\pi_0(x' \,|\, z)} \underbrace{\mu_0(x', z)}_{=\mathbb{E}[Y|x,z]} \underbrace{P(x' \,|\, z)}_{=\pi_0(x'|z)} P(z)$$

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Derivation of Doubly Robust Estimand - 2

$$C(P) = \sum_z \mathbb{E}[Y \mid x, z] P(z)$$

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$$= C(P)$$

Derivation of Doubly Robust Estimand - 3

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$$C(P) = \mathbb{E}\left[f^{IPW}(V; \pi_0)\right] + \mathbb{E}[f^{REG}(V; \mu_0)] - \mathbb{E}\left[\frac{I_x(X)}{\pi_0(X \mid Z)} \mu_0(X, Z)\right]$$

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$$= \mathbb{E}\left[\frac{I_x(X)}{\pi_0(X|Z)}\left\{Y - \mu_0(X,Z)\right\} + \mu_0(x,Z)\right]$$

Derivation of Doubly Robust Estimand - 3

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$$= \mathbb{E}\left[\frac{I_x(X)}{\pi_0(X|Z)}\left\{Y - \mu_0(X,Z)\right\} + \mu_0(x,Z)\right]$$

$$f^{DR}(V;\{\pi,\mu\}) := \frac{I_x(X)}{\pi(X|Z)}\{Y - \mu(X,Z)\} + \mu(x,Z)$$

Doubly Robustness - Proof 1

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$\mathbb{E}[f^{DR}(V; \{\pi_0, \mu\})] = C(P)$ for any μ

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Doubly Robustness - Proof 2

$$\mathbb{E}[f^{DR}(V; \eta = \{\pi_0, \mu\})] = C(P) \text{ for any } \mu$$

$$\mathbb{E}[f^{DR}(V; \eta = \{\pi_0, \mu\})]$$
$$= \mathbb{E}\left[f^{IPW}(V; \pi_0)\right] \text{ ——— A}$$
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$$\mathbb{E}[f^{DR}(V;\eta=\{\pi_0,\mu\})] = C(P) \text{ for any } \mu$$

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$$= \sum_z \mu(x,z)P(z) = \mathbb{E}[\mu(x,Z)]$$

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$$= \sum_{z} \mu(x, z)P(z) = \mathbb{E}[\mu(x, Z)]$$

$$= \mathbb{E}[f^{REG}(V; \mu)]$$

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$$= \sum_z \mu(x,z)P(z) = \mathbb{E}[\mu(x,Z)]$$

$$= \mathbb{E}[f^{REG}(V;\mu)]$$

$$= B$$

$$\mathbb{E}[f^{DR}(V;\eta=\{\pi_0,\mu\})]$$

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$$\mathbb{E}[f^{DR}(V; \eta = \{\pi_0, \mu\})]$$

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$$= C(P)$$

Doubly Robustness - Proof 3

$\mathbb{E}[f^{DR}(V; \eta = \{\pi, \mu_0\})] = C(P)$ for any positive π

$$\mathbb{E}[f^{DR}(V; \eta = \{\pi, \mu_0\})]$$
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Doubly Robustness - Proof 3

$\mathbb{E}[f^{DR}(V; \eta = \{\pi, \mu_0\})] = C(P)$ for any positive π

$$B = \mathbb{E}[f^{REG}(V; \mu_0)] = C(P)$$

$$\mathbb{E}[f^{DR}(V; \eta = \{\pi, \mu_0\})]$$

$$= \mathbb{E}\left[f^{IPW}(V; \pi)\right] \quad \text{——— A}$$

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Doubly Robustness - Proof 4

$\mathbb{E}[f^{DR}(V;\eta=\{\pi,\mu_0\})]=C(P)$ for any positive π

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$$\mathbb{E}[f^{DR}(V;\eta=\{\pi,\mu_0\})]$$

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$$\mathbb{E}[f^{DR}(V; \eta = \{\pi, \mu_0\})]$$
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$\mathbb{E}[f^{DR}(V;\eta=\{\pi,\mu_0\})]=C(P)$ for any positive π

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$$=\mathbb{E}\left[\frac{I_x(X)}{\pi(X|Z)}Y\right]=\mathbb{E}[f^{IPW}(V;\pi)]$$

$$=A$$

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$$=A+C(P)-A$$

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Intermediate Summary - Estimands

So far…

1. We defined an estimand $f(V; \eta)$, a functional for estimating the causal effects.

2. For the BD adjustment, we illustrated three estimands (REG, IPW, DR), and showed that the DR estimand has the doubly-robustness property.

Next, we will introduce a general principle for choosing an estimand.

Orthogonal Estimand

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If $\mathbb{E}[f(V;\eta)]$ is invariant to the small perturbation of η,

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Idea of an Orthogonal Estimand

If $\mathbb{E}[f(V;\eta)]$ is invariant to the small perturbation of η,

… despite the error of η,

… $\mathbb{E}[f(V;\eta)]$ will not suffer from the error.

We will formalize this idea by considering the *directional derivative* of $\mathbb{E}[f(V;\eta)]$.

Directional Derivative & Orthogonal Estimand

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Directional Derivative:

For a function $g(\eta)$, its derivative along the direction h at η_0 is given as

$$D_\eta g(\eta_0)\{h\} := \frac{\partial}{\partial t} g(\eta_0 + th)\Big|_{t=0}$$

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Orthogonal Estimand:

$f(V;\eta)$ is an *orthogonal estimand* if

$$D_\eta \mathbb{E}[f(V;\eta_0)]\{\eta - \eta_0\} = 0$$

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Directional Derivative:

For a function $g(\eta)$, its derivative along the direction h at η_0 is given as

$$D_\eta g(\eta_0)\{h\} := \frac{\partial}{\partial t} g(\eta_0 + th)\bigg|_{t=0}$$

Orthogonal Estimand:

$f(V;\eta)$ is an *orthogonal estimand* if

$$D_\eta \mathbb{E}[f(V;\eta_0)]\{\eta - \eta_0\} = 0$$

The estimand is invariant along the error $(\eta - \eta_0)$ at the true nuisance η_0

Debiasedness of Orthogonal Estimands

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For any η, $\mathbb{E}[f(V;\eta)] - C(P) = O_P\left(\| \eta - \eta_0 \|_2^2\right)$

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$\mathbb{E}[f(V;\eta)]$ converges to $C(P)$ at $(N^{-1/4})^2 = N^{-1/2}$ rate.

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Then, by Taylor's expansion (up to the 2nd order),

$$\mathbb{E}[f(V;\eta)] - \mathbb{E}[f(V;\eta_0)] = D_\eta \mathbb{E}[f(V;\eta_0)]\{\eta - \eta_0\} + \frac{1}{2} D_\eta^2 \mathbb{E}[f(V;\eta)]\{\eta - \eta_0\}^2$$

Debiasedness of Orthogonal Estimands - Proof

$$D_\eta \mathbb{E}[f(V;\eta_0)]\{\eta - \eta_0\} = 0$$

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For any η, $\mathbb{E}[f(V;\eta)] - C(P) = O_P\left(\| \eta - \eta_0 \|_2^2\right)$

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0

Debiasedness of Orthogonal Estimands - Proof

$$D_\eta \mathbb{E}[f(V;\eta_0)]\{\eta-\eta_0\} = 0$$

Debiasedness

For any η, $\mathbb{E}[f(V;\eta)] - C(P) = O_P\left(\| \eta - \eta_0 \|_2^2\right)$

Then, by Taylor's expansion (up to the 2nd order),

$$\mathbb{E}[f(V;\eta)] - \mathbb{E}[f(V;\eta_0)] = \underbrace{D_\eta \mathbb{E}[f(V;\eta_0)]\{\eta-\eta_0\}}_{0} + \frac{1}{2} D_\eta^2 \mathbb{E}[f(V;\eta)]\{\eta-\eta_0\}^2$$

$O_P\left(\| \eta - \eta_0 \|_{L_2(P)}^2\right) := O_P\left(\mathbb{E}[(\eta-\eta_0)^2]\right)$, shortly, $O_P\left(\| \eta - \eta_0 \|^2\right)$.

Orthogonal Estimand - Two nuisances

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$$\mathbb{E}[f(V;\{\eta^a,\eta^b\})] - C(P)$$
$$= O_P\left(\| \eta^a - \eta_0^a \|^2\right) + O_P\left(\| \eta^b - \eta_0^b \|^2\right) + O_P\left(\| \eta^a - \eta_0^a \| \| \eta^b - \eta_0^b \|\right)$$

Orthogonal Estimand - Two nuisances

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If $\eta = \{\eta^a, \eta^b\}$ converges to $\eta_0 := \{\eta^a_0, \eta^b_0\}$ at some rate, say $N^{-1/4}$

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Orthogonal Estimand - Two nuisances (Proof)

$$\mathbb{E}[f(V;\{\eta^a,\eta^b\})] - C(P)$$
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$$+\frac{1}{2}D^2_{\eta^b}\mathbb{E}[f(V;\{\eta^a,\eta^b\})]\{\eta^b - \eta_0^b\}^2$$
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Orthogonal Estimand - Two nuisances (Proof)

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Orthogonal Estimand - Two nuisances (Proof)

$$\mathbb{E}[f(V;\{\eta^a,\eta^b\})] - C(P)$$
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$$\mathbb{E}[f(V;\{\eta^a,\eta^b\})] - \mathbb{E}[f(V;\{\eta_0^a.\eta_0^b\})]$$
$$= D_{\eta^a}\mathbb{E}[f(V;\{\eta_0^a,\eta_0^b\})]\{\eta^a-\eta_0^a\}$$
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$$+\frac{1}{2}D^2_{\eta^b}\mathbb{E}[f(V;\{\eta^a,\eta^b\})]\{\eta^b-\eta_0^b\}^2$$
$$+D_{\eta_a}D_{\eta_b}\mathbb{E}[f(V;\{\eta^a,\eta^b\}]\{\eta^a-\eta_0^a,\eta^b-\eta_0^b\}$$

0

First-order derivative

Orthogonal Estimand - Two nuisances (Proof)

$$\mathbb{E}[f(V;\{\eta^a,\eta^b\})] - C(P)$$
$$= O_P\left(\| \eta^a - \eta_0^a \|^2\right) + O_P\left(\| \eta^b - \eta_0^b \|^2\right) + O_P\left(\| \eta^a - \eta_0^a \|\| \eta^b - \eta_0^b \|\right)$$

$$\mathbb{E}[f(V;\{\eta^a,\eta^b\})] - \mathbb{E}[f(V;\{\eta_0^a . \eta_0^b\})]$$
$$= D_{\eta^a}\mathbb{E}[f(V;\{\eta_0^a,\eta_0^b\})]\{\eta^a - \eta_0^a\}$$
$$+ D_{\eta^b}\mathbb{E}[f(V;\{\eta_0^a,\eta_0^b\})]\{\eta^b - \eta_0^b\}$$

First-order derivative

0

$$+ \frac{1}{2}D^2_{\eta^a}\mathbb{E}[f(V;\{\eta^a,\eta^b\})]\{\eta - \eta_0\}^2$$

$$= O_P\left(\| \eta^a - \eta_0^a \|^2\right)$$

$$+ \frac{1}{2}D^2_{\eta^b}\mathbb{E}[f(V;\{\eta^a,\eta^b\})]\{\eta^b - \eta_0^b\}^2$$
$$+ D_{\eta_a}D_{\eta_b}\mathbb{E}[f(V;\{\eta^a,\eta^b\}]\{\eta^a - \eta_0^a, \eta^b - \eta_0^b\}$$

Orthogonal Estimand - Two nuisances (Proof)

$$\mathbb{E}[f(V;\{\eta^a,\eta^b\})] - C(P)$$
$$= O_P\left(\| \eta^a - \eta_0^a \|^2\right) + O_P\left(\| \eta^b - \eta_0^b \|^2\right) + O_P\left(\| \eta^a - \eta_0^a \| \| \eta^b - \eta_0^b \|\right)$$

$$\mathbb{E}[f(V;\{\eta^a,\eta^b\})] - \mathbb{E}[f(V;\{\eta_0^a . \eta_0^b\})]$$
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0

First-order derivative

$$= O_P\left(\| \eta^a - \eta_0^a \|^2\right)$$

$$= O_P\left(\| \eta^b - \eta_0^b \|^2\right)$$

Orthogonal Estimand - Two nuisances (Proof)

$$\mathbb{E}[f(V;\{\eta^a,\eta^b\})] - C(P)$$
$$= O_P\left(\|\eta^a-\eta_0^a\|^2\right) + O_P\left(\|\eta^b-\eta_0^b\|^2\right) + O_P\left(\|\eta^a-\eta_0^a\|\|\eta^b-\eta_0^b\|\right)$$

$$\mathbb{E}[f(V;\{\eta^a,\eta^b\})] - \mathbb{E}[f(V;\{\eta_0^a.\eta_0^b\})]$$
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$$+D_{\eta^b}\mathbb{E}[f(V;\{\eta_0^a,\eta_0^b\})]\{\eta^b-\eta_0^b\}$$

First-order derivative

0

$$+\frac{1}{2}D^2_{\eta^a}\mathbb{E}[f(V;\{\eta^a,\eta^b\})]\{\eta-\eta_0\}^2 \quad = O_P\left(\|\eta^a-\eta_0^a\|^2\right)$$
$$+\frac{1}{2}D^2_{\eta^b}\mathbb{E}[f(V;\{\eta^a,\eta^b\})]\{\eta^b-\eta_0^b\}^2 \quad = O_P\left(\|\eta^b-\eta_0^b\|^2\right)$$
$$+D_{\eta_a}D_{\eta_b}\mathbb{E}[f(V;\{\eta^a,\eta^b\}]\{\eta^a-\eta_0^a,\eta^b-\eta_0^b\} \quad = O_P\left(\|\eta^a-\eta_0^a\|\|\eta^b-\eta_0^b\|\right)$$

Is the REG orthogonal?

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$f^{REG}(V; \mu)$ estimand is non-orthogonal.

Is the REG orthogonal?

$f^{REG}(V;\mu)$ estimand is non-orthogonal.

$$
\begin{aligned}
D_\mu \mathbb{E}[f^{REG}(V;\mu_0)]\{\mu-\mu_0\} &:= \frac{\partial}{\partial t}\mathbb{E}\left[f^{REG}(V;\mu+t(\mu-\mu_0)\right]|_{t=0} \\
&= \mathbb{E}\left[\frac{\partial}{\partial t} f^{REG}(V;\mu+t(\mu-\mu_0)\,|_{t=0}\right] \\
&= \mathbb{E}\left[\frac{\partial}{\partial t}\{\mu+t(\mu-\mu_0)\}\,|_{t=0}\right] \\
&= \mathbb{E}[\mu(x,Z)-\mu_0(x,Z)]
\end{aligned}
$$

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\begin{aligned}
D_\mu \mathbb{E}[f^{REG}(V;\mu_0)]\{\mu-\mu_0\} &:= \frac{\partial}{\partial t}\mathbb{E}\left[f^{REG}(V;\mu+t(\mu-\mu_0)\right] |_{t=0} \\
&= \mathbb{E}\left[\frac{\partial}{\partial t} f^{REG}(V;\mu+t(\mu-\mu_0)\,|_{t=0}\right] \\
&= \mathbb{E}\left[\frac{\partial}{\partial t}\{\mu+t(\mu-\mu_0)\}\,|_{t=0}\right] \\
&= \mathbb{E}[\mu(x,Z)-\mu_0(x,Z)] \\
&\neq 0
\end{aligned}
$$

Is the IPW orthogonal?

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$f^{IPW}(V;\pi)$ estimand is non-orthogonal.

Is the IPW orthogonal?

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$$D_\pi \mathbb{E}[f^{IPW}(V;\pi_0)]\{\pi-\pi_0\} := \frac{\partial}{\partial t}\mathbb{E}\left[f^{IPW}(V;\pi+t(\pi-\pi_0)\right]|_{t=0}$$

$$= \mathbb{E}\left[\frac{\partial}{\partial t}f^{IPW}(V;\pi+t(\pi-\pi_0)\,|_{t=0}\right]$$

$$= \mathbb{E}\left[(\pi-\pi_0)\frac{\partial}{\partial \pi}f^{IPW}(V;\pi)\,|_{\pi=\pi_0}\right]$$

$$= -\,\mathbb{E}\left[\{\pi-\pi_0\}\left\{\frac{I_x(X)}{\pi_0^2(X\,|\,Z)}Y\right\}\right]$$

Is the IPW orthogonal?

$f^{IPW}(V;\pi)$ estimand is non-orthogonal.

$$D_\pi \mathbb{E}[f^{IPW}(V;\pi_0)]\{\pi - \pi_0\} := \frac{\partial}{\partial t}\mathbb{E}\left[f^{IPW}(V;\pi + t(\pi - \pi_0)\right]|_{t=0}$$

$$= \mathbb{E}\left[\frac{\partial}{\partial t} f^{IPW}(V;\pi + t(\pi - \pi_0)\,|_{t=0}\right]$$

$$= \mathbb{E}\left[(\pi - \pi_0)\frac{\partial}{\partial \pi} f^{IPW}(V;\pi)\,|_{\pi=\pi_0}\right]$$

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$$\neq 0$$

Is the DR estimand orthogonal? - 1

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$f^{DR}(V; \{\pi, \mu_0\})$ is an orthogonal estimand

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$f^{DR}(V; \{\pi, \mu_0\})$ is an orthogonal estimand

$$D_\mu \mathbb{E}[f^{DR}(V; \{\pi, \mu_0\})]\{\mu - \mu_0\} := \frac{\partial}{\partial t} \mathbb{E}\left[f^{DR}(V; \{\pi_0, \mu + t(\mu - \mu_0)\}\right] |_{t=0}$$

$$= \mathbb{E}\left[\frac{\partial}{\partial t} f^{DR}(V; \{\pi_0, \mu + t(\mu - \mu_0)\}) |_{t=0}\right]$$

$$= \mathbb{E}\left[(\mu - \mu_0) \frac{\partial}{\partial \mu} f^{DR}(V; \{\pi, \mu\}) |_{\mu=\mu_0}\right]$$

$$= \mathbb{E}\left[\{\mu - \mu_0\}\left\{-\frac{I_x(X)}{\pi_0(X|Z)}\mu_0(X, Z) + \mu_0(x, Z)\right\}\right]$$

$$= \mathbb{E}\left[\{\mu - \mu_0\}\left\{-\mu_0(x, Z) + \mu_0(x, Z)\right\}\right]$$

Is the DR estimand orthogonal? - 1

$f^{DR}(V;\{\pi,\mu_0\})$ is an orthogonal estimand

$$
\begin{aligned}
D_\mu \mathbb{E}[f^{DR}(V;\{\pi,\mu_0\})]\{\mu-\mu_0\} := & \frac{\partial}{\partial t}\mathbb{E}\left[f^{DR}(V;\{\pi_0,\mu+t(\mu-\mu_0)\}\right]|_{t=0} \\
&= \mathbb{E}\left[\frac{\partial}{\partial t} f^{DR}(V;\{\pi_0,\mu+t(\mu-\mu_0)\})|_{t=0}\right] \\
&= \mathbb{E}\left[(\mu-\mu_0)\frac{\partial}{\partial \mu} f^{DR}(V;\{\pi,\mu\})|_{\mu=\mu_0}\right] \\
&= \mathbb{E}\left[\{\mu-\mu_0\}\left\{-\frac{I_x(X)}{\pi_0(X|Z)}\mu_0(X,Z)+\mu_0(x,Z)\right\}\right] \\
&= \mathbb{E}\left[\{\mu-\mu_0\}\{-\mu_0(x,Z)+\mu_0(x,Z)\}\right]
\end{aligned}
$$

0

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$f^{DR}(V; \{\pi, \mu_0\})$ is an orthogonal estimand

$$
\begin{aligned}
D_\mu \mathbb{E}[f^{DR}(V; \{\pi, \mu_0\})]\{\mu - \mu_0\} &:= \frac{\partial}{\partial t} \mathbb{E}\left[f^{DR}(V; \{\pi_0, \mu + t(\mu - \mu_0)\}\right] |_{t=0} \\
&= \mathbb{E}\left[\frac{\partial}{\partial t} f^{DR}(V; \{\pi_0, \mu + t(\mu - \mu_0)\}) |_{t=0}\right] \\
&= \mathbb{E}\left[(\mu - \mu_0)\frac{\partial}{\partial \mu} f^{DR}(V; \{\pi, \mu\}) |_{\mu=\mu_0}\right] \\
&= \mathbb{E}\left[\{\mu - \mu_0\}\left\{-\frac{I_x(X)}{\pi_0(X|Z)}\mu_0(X, Z) + \mu_0(x, Z)\right\}\right] \\
&= \mathbb{E}\left[\{\mu - \mu_0\}\{-\mu_0(x, Z) + \mu_0(x, Z)\}\right] \\
&= 0
\end{aligned}
$$

0

Is the DR estimand orthogonal? - 2

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\begin{aligned}
D_\pi \mathbb{E}[f^{DR}(V; \{\pi, \mu_0\})]\{\pi - \pi_0\} := & \frac{\partial}{\partial t}\mathbb{E}\left[f^{DR}(V; \{\pi + t(\pi - \pi_0), \mu_0\}\right] |_{t=0} \\
&= \mathbb{E}\left[\frac{\partial}{\partial t} f^{DR}(V; \{\pi + t(\pi - \pi_0), \mu_0\}) |_{t=0}\right] \\
&= \mathbb{E}\left[(\pi - \pi_0)\frac{\partial}{\partial \pi} f^{DR}(V; \{\pi, \mu_0\}) |_{\pi=\pi_0}\right] \\
&= \mathbb{E}\left[\{\pi - \pi_0\}\left\{-\frac{I_x(X)}{\pi_0^2(X|Z)}\{Y - \mu_0(X, Z)\}\right\}\right] \\
&= \mathbb{E}\left[\{\pi - \pi_0\}\left\{-\frac{I_x(X)}{\pi_0^2(X|Z)}\{\mu_0(X, Z) - \mu_0(X, Z)\}\right\}\right]
\end{aligned}
$$

Is the DR estimand orthogonal? - 2

$f^{DR}(V; \{\pi, \mu_0\})$ is an orthogonal estimand

$$D_\pi \mathbb{E}[f^{DR}(V; \{\pi, \mu_0\})]\{\pi - \pi_0\} := \frac{\partial}{\partial t}\mathbb{E}\left[f^{DR}(V; \{\pi + t(\pi - \pi_0), \mu_0\}\right] |_{t=0}$$

$$= \mathbb{E}\left[\frac{\partial}{\partial t} f^{DR}(V; \{\pi + t(\pi - \pi_0), \mu_0\}) |_{t=0}\right]$$

$$= \mathbb{E}\left[(\pi - \pi_0)\frac{\partial}{\partial \pi} f^{DR}(V; \{\pi, \mu_0\}) |_{\pi=\pi_0}\right]$$

$$= \mathbb{E}\left[\{\pi - \pi_0\}\left\{-\frac{I_x(X)}{\pi_0^2(X|Z)}\{Y - \mu_0(X, Z)\}\right\}\right]$$

$$= \mathbb{E}\left[\{\pi - \pi_0\}\left\{-\frac{I_x(X)}{\pi_0^2(X|Z)}\{\mu_0(X, Z) - \mu_0(X, Z)\}\right\}\right]$$

0

Is the DR estimand orthogonal? - 2

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$$
\begin{aligned}
D_\pi \mathbb{E}[f^{DR}(V;\{\pi,\mu_0\})]\{\pi-\pi_0\} &:= \frac{\partial}{\partial t}\mathbb{E}\left[f^{DR}(V;\{\pi+t(\pi-\pi_0),\mu_0\}\right]|_{t=0} \\
&= \mathbb{E}\left[\frac{\partial}{\partial t} f^{DR}(V;\{\pi+t(\pi-\pi_0),\mu_0\})\,|_{t=0}\right] \\
&= \mathbb{E}\left[(\pi-\pi_0)\frac{\partial}{\partial \pi} f^{DR}(V;\{\pi,\mu_0\})\,|_{\pi=\pi_0}\right] \\
&= \mathbb{E}\left[\{\pi-\pi_0\}\left\{-\frac{I_x(X)}{\pi_0^2(X|Z)}\{Y-\mu_0(X,Z)\}\right\}\right] \\
&= \mathbb{E}\left[\{\pi-\pi_0\}\left\{-\frac{I_x(X)}{\pi_0^2(X|Z)}\{\mu_0(X,Z)-\mu_0(X,Z)\}\right\}\right] \quad 0 \\
&= 0
\end{aligned}
$$

Debiasedness and Doubly Robustness

Debiasedness:

If $f(V;\eta)$ is *orthogonal*, $\mathbb{E}[f(V;\eta)] - C(P) = O_P\left(\| \eta - \eta_0 \|_2^2\right)$

Converging at $N^{-1/2}$ if π, μ converges at $N^{-1/4}$ ("*debiasedness*")

Debiasedness and Doubly Robustness

Debiasedness:

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No errors if either $\pi = \pi_0$ or $\mu = \mu_0$ ("doubly-robustness")

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Next, we will study finite sample properties.

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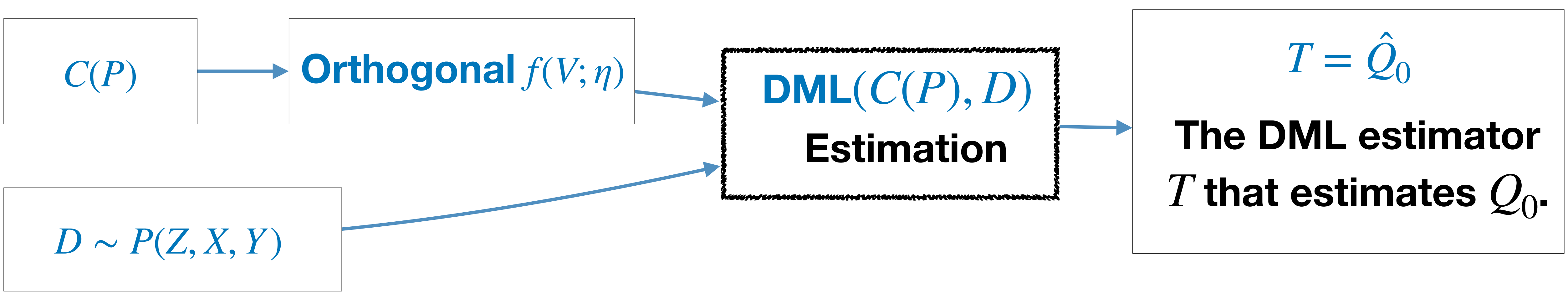

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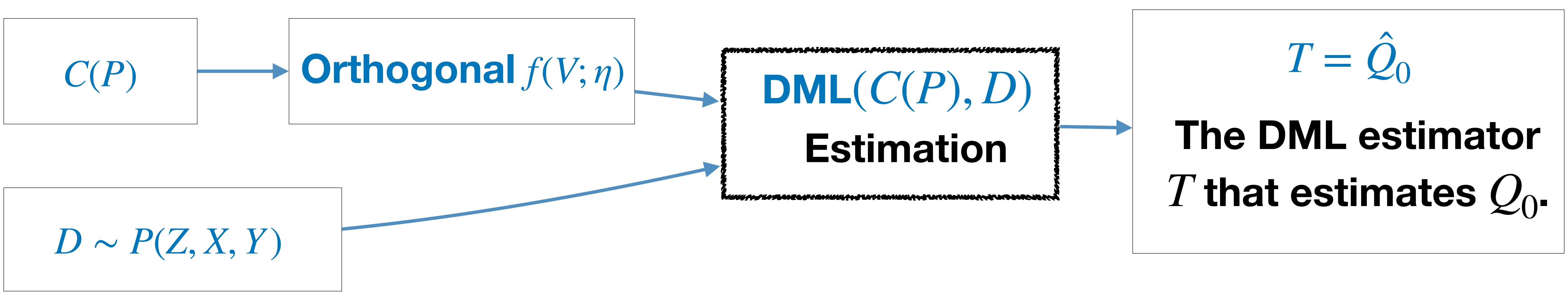

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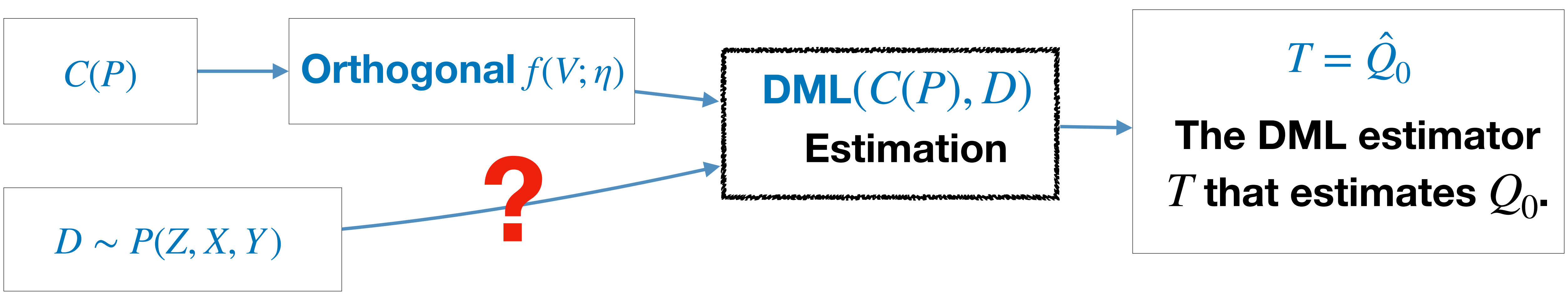
Now, we connect the estimand to the estimation task using finite samples.



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We will focus on analyzing the remaining term: $\mathbb{E}_D[f(V;\hat{\eta})] - \mathbb{E}_P[f(V;\hat{\eta})]$.

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Concentration inequalities (e.g., Hoeffding's inequality)

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… Without any special treatises, $\mathbb{E}_D[f(V;\hat{\eta})] - \mathbb{E}_P[f(V;\hat{\eta})]$ is *not necessarily converging to 0.*

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Then, $\mathbb{E}_D[f(V;\hat{\eta})] - \mathbb{E}_P[f(V;\hat{\eta})] \to 0$ obviously holds.

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A class H s.t. $\sup_{\eta \in H} \left(\mathbb{E}_D[f(V;\eta)] - \mathbb{E}_P[f(V;\eta)] \right) \to 0$ at $N^{-1/2}$ rate

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: differentiable functions

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Then, the estimator $\mathbb{E}_D[f(V;\hat{\eta})]$ *converges to* $C(P)$ *fast even if* $\hat{\eta}$ *converges slow*

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This doesn't require the Donsker class assumption!

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Any ML models can be employed!

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Now we are ready to formally define the DML estimator.

So, what’s the DML?

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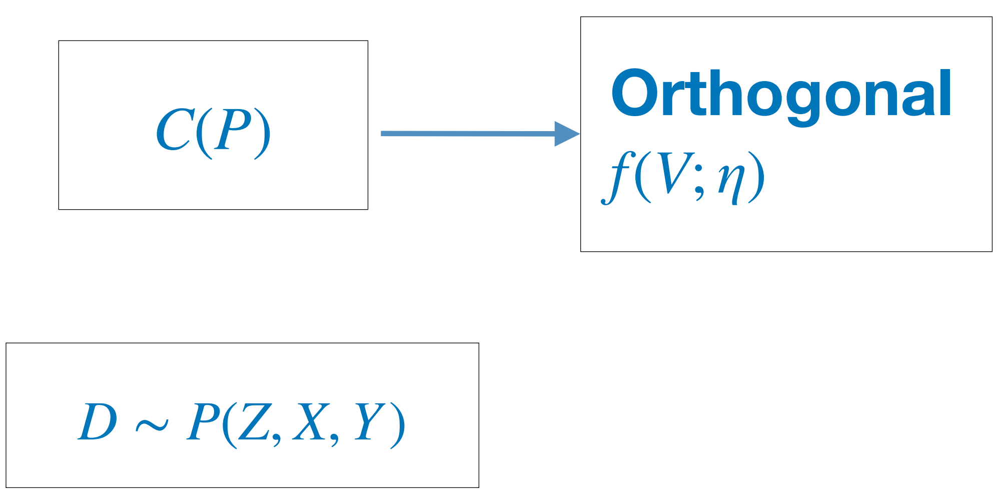

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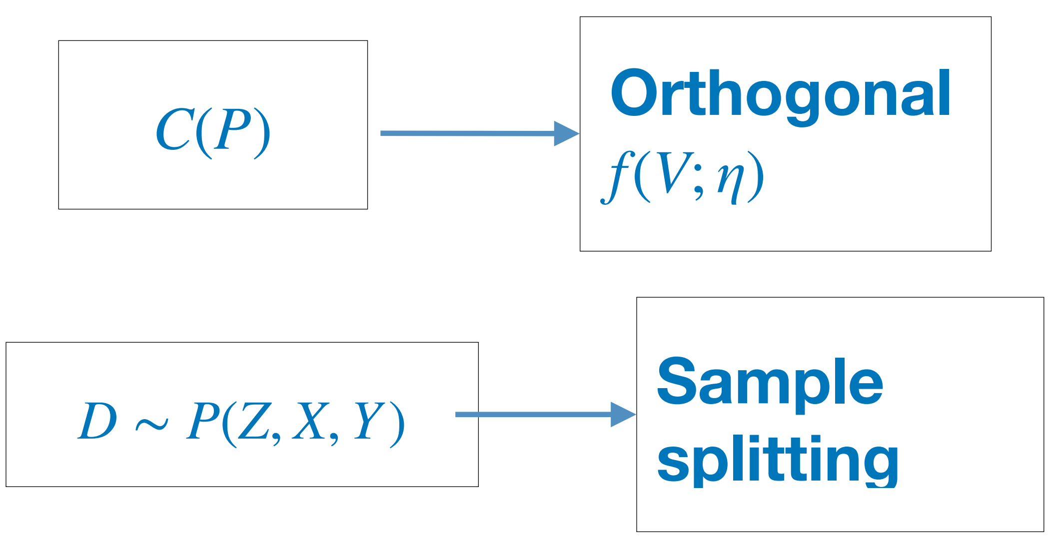

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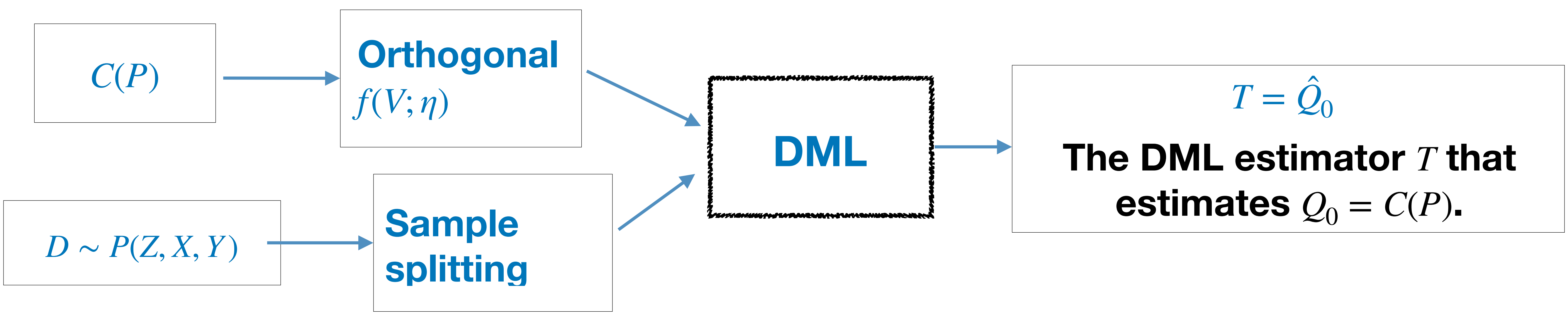

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Any Questions?

Estimating Causal Effects

Yonghan Jung
Purdue University
yonghanjung.me

2022.07.12

University of Seoul

Introduction

Outline

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- **Part 1**: Double/Debiased Machine Learning (DML)

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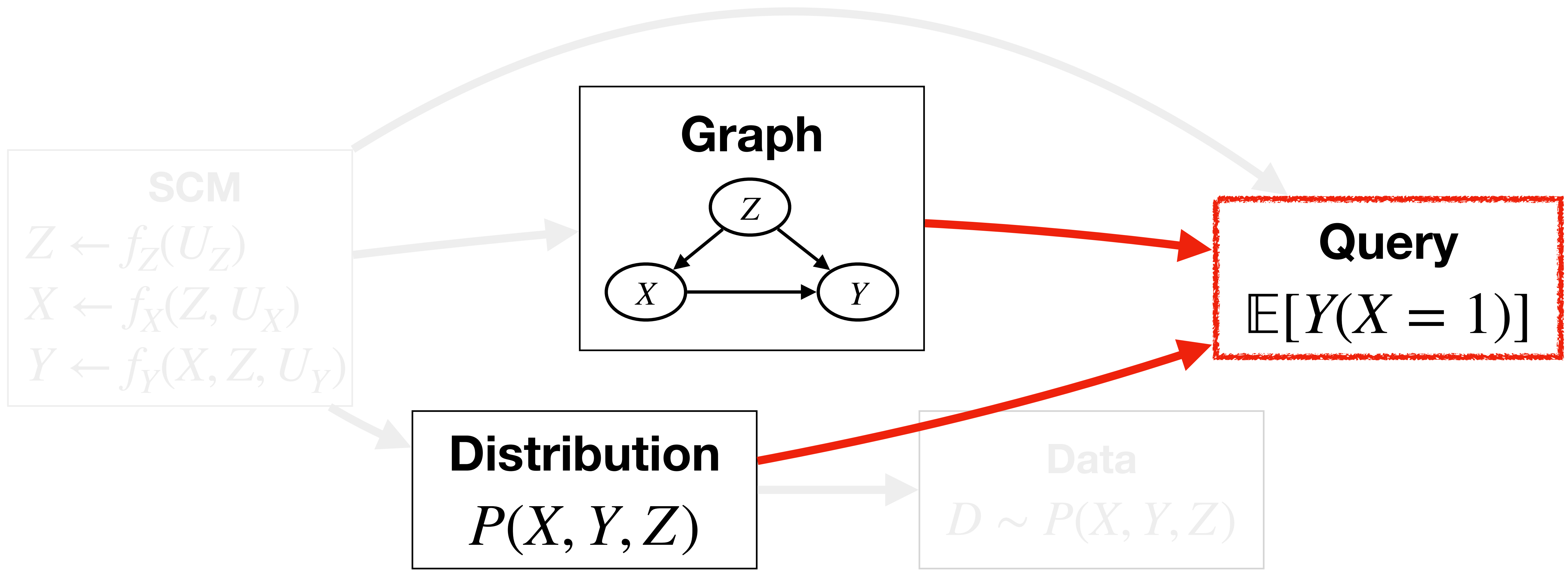
- **Part 2**: DML for any identifiable functional

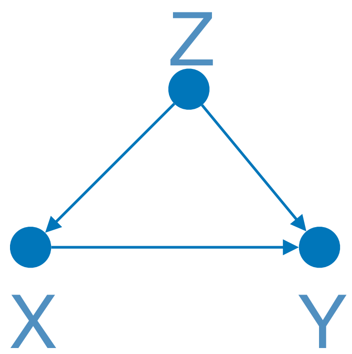
  Jung, Tian, Bareinboim (2021a). Estimating Identifiable Causal Effects through Double Machine Learning. In Proceedings of the 35th AAAI Conference on AI, 2021.

Part II.
Double ML-ID

1. Causal Effect Identification: Big Picture (1)



Causal Effect Identification



Causal graph (G)

$$P(Z, X, Y)$$

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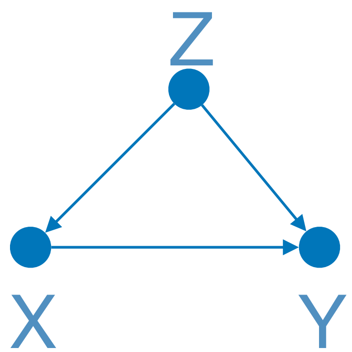
$$Q_0 := \mathbb{E}[Y \mid do(x)]$$

Causal Query (Q_0)

Causal Effect Identification



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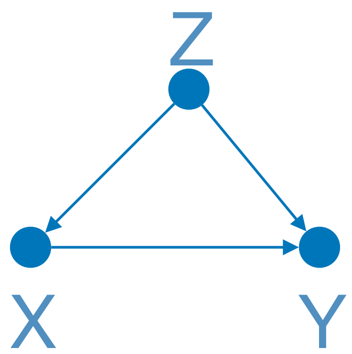

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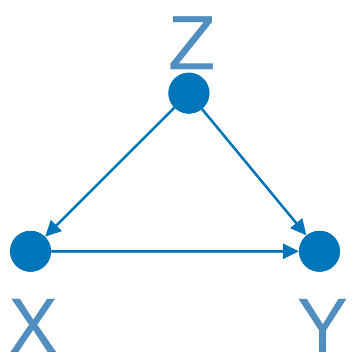
finds the functional $C(P)$ s.t. $C(P) = Q_0$.



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Causal functional

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↳ $T(\eta_1, \eta_2)$ converges fast even when either η_1 or η_2 is correct; or η_1, η_2 converge slowly.

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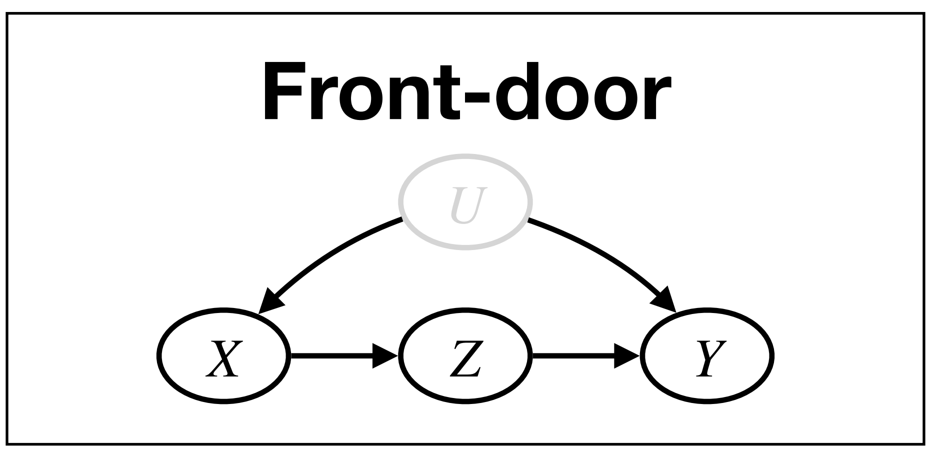

$$\mathbb{E}[Y(x)] = \sum_{z} P(z \mid x) \sum_{x'} \mathbb{E}[Y \mid x', z] P(x')$$

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$P(\mathbf{v}) > 0$ for any v.

Multi-outcome sequential BD (mSBD)

Jung, Yonghan, Jin Tian, and Elias Bareinboim. "Estimating causal effects using weighting-based estimators." Proceedings of the AAAI Conference on Artificial Intelligence. Vol. 34. No. 06. 2020.

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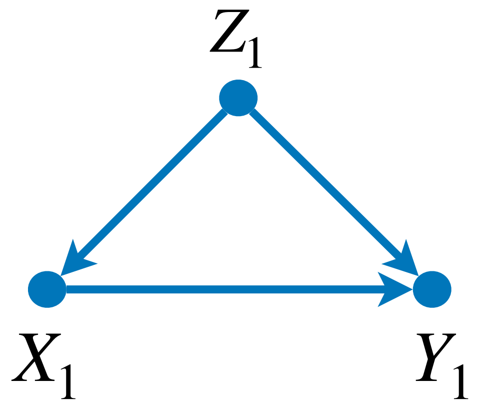

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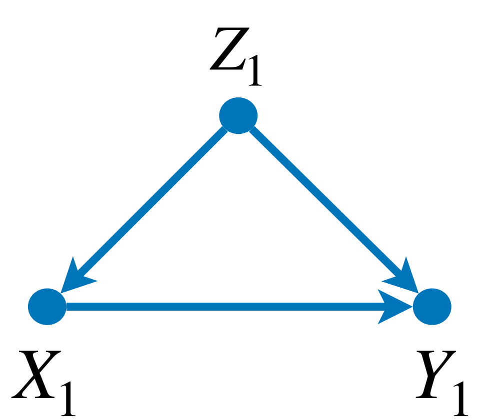

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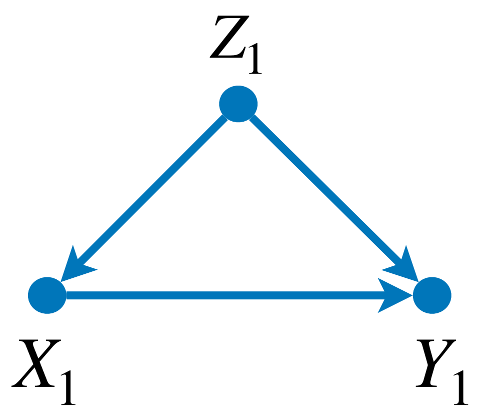

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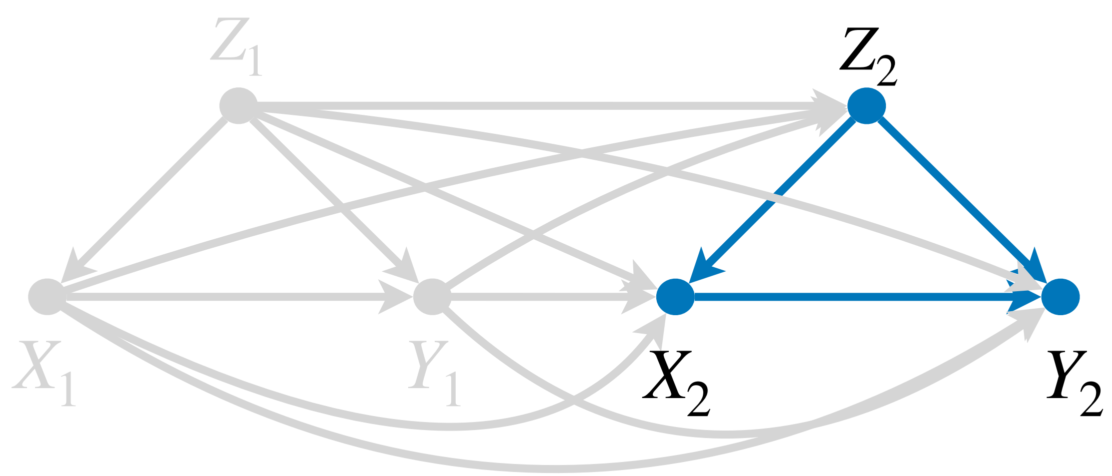

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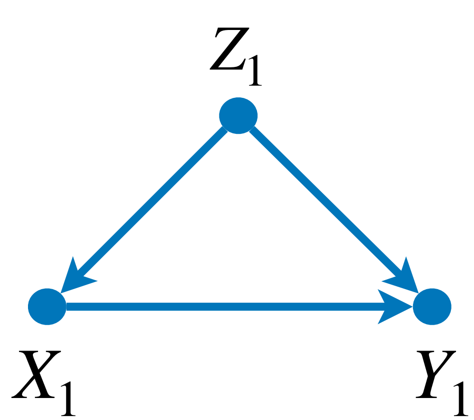

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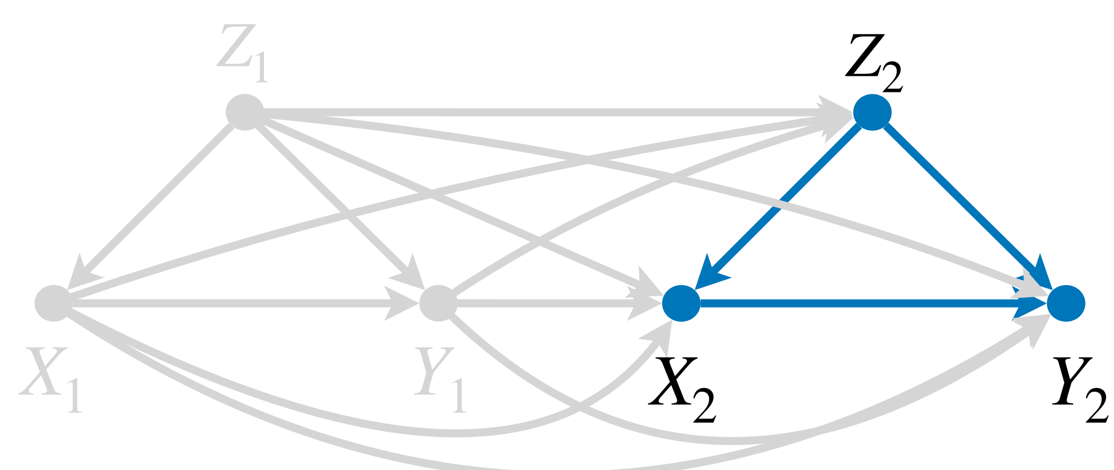

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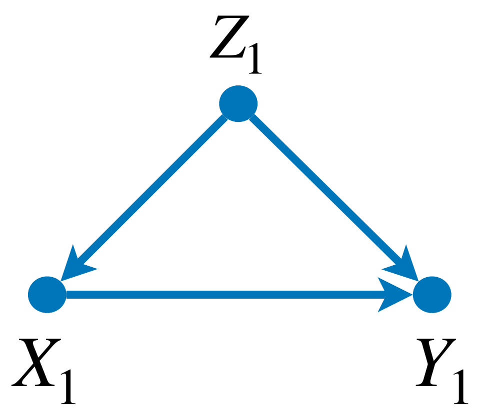

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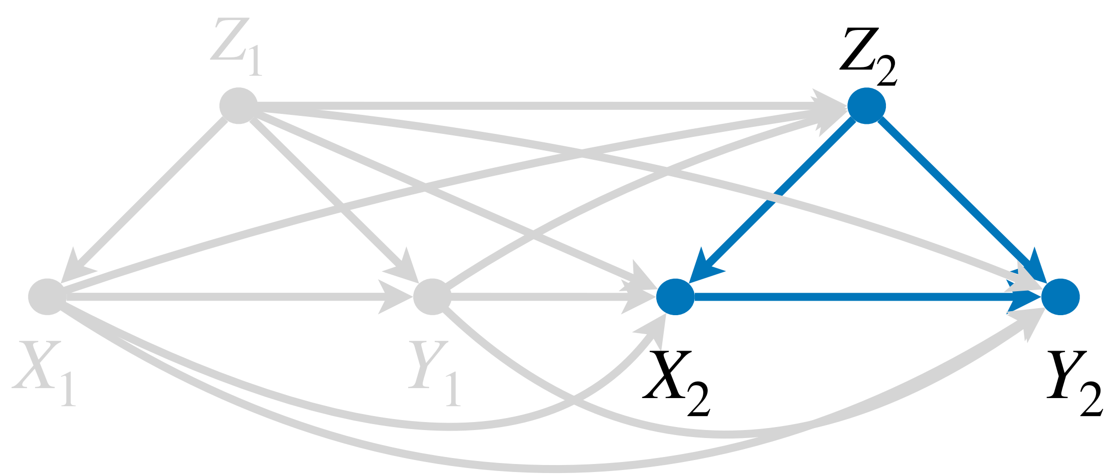

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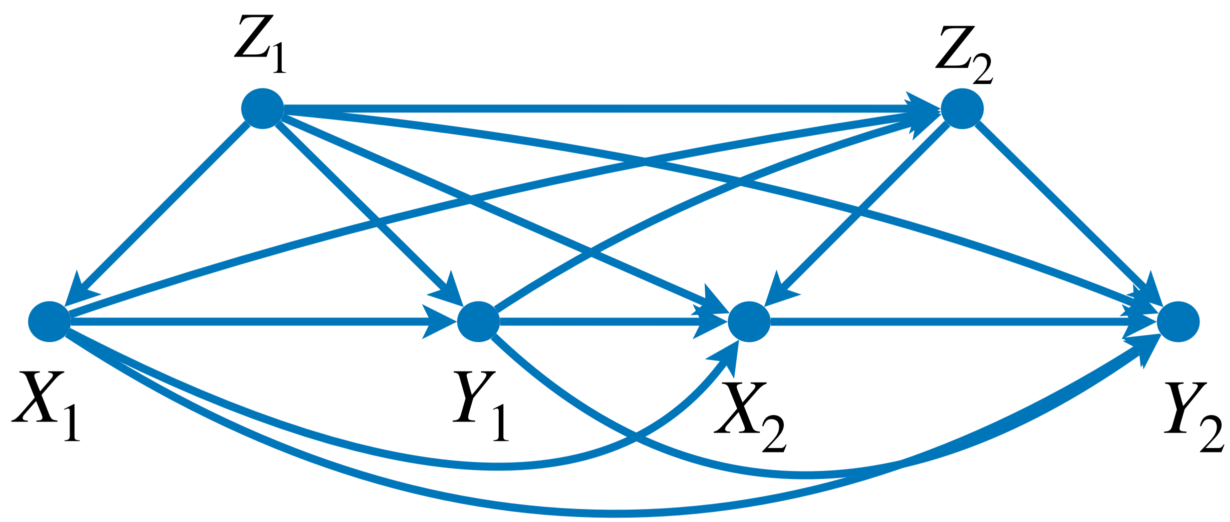

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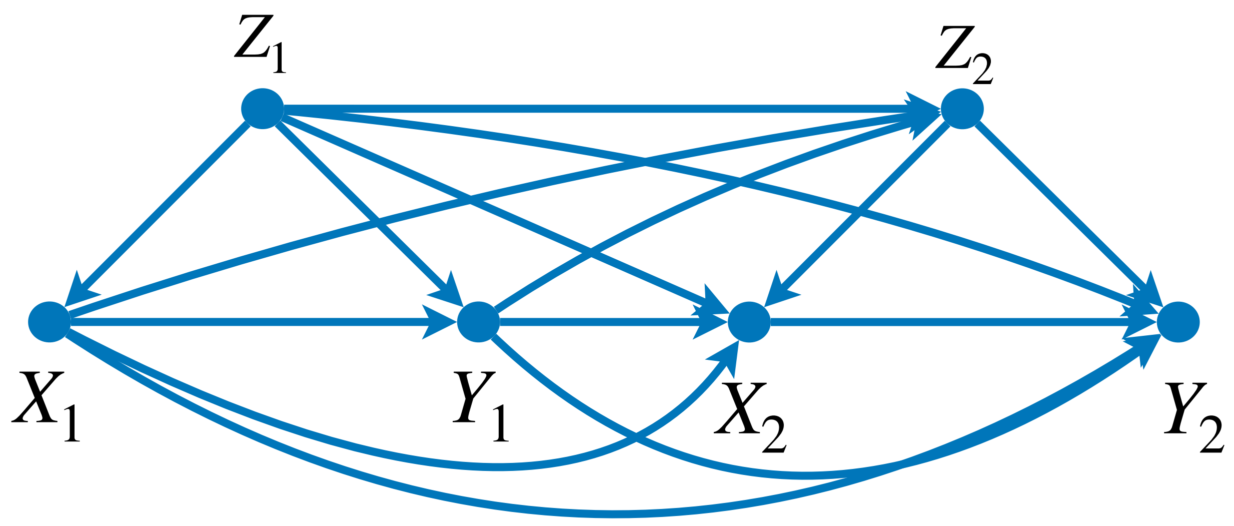

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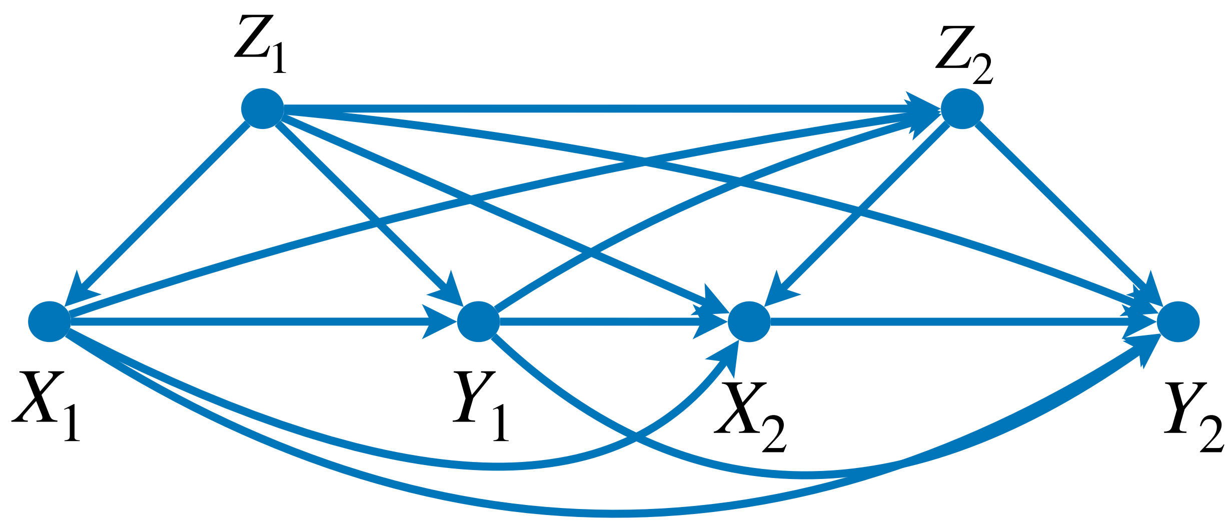

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REG Estimand for mSBD - Proof

$$\mathbb{E}[f^{REG}(\mathbf{V};\mathbf{H}_0)] = \mathbb{E}[H_0^1(x_1)] = C(P)$$

$$P_{x_1,x_2}(y_1,y_2) = \sum_{z_1,z_2} P(z_1)P(y_1 \mid x_1,z_1)P(z_2 \mid z_1,x_1,y_1)P(y_2 \mid z_1,x_1,y_1,z_2,x_2)$$

$$H_0^3(x_3) := I_{y_1,y_2}(Y_1,Y_2)$$

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$$H_0^2(x_2) = \mathbb{E}[H_0^3 \mid x_2, X_1, Y_1, \mathbf{Z}^{(2)}] = I_{y_1}(Y_1)P(y_2 \mid x_2, X_1, Y_1, \mathbf{Z}^{(2)})$$

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$$H_0^1(x_1) = \mathbb{E}[H_0^2 \mid x_1, Z_1] = \sum_{z_2} P(y_2 \mid \mathbf{x}^{(2)}, y_1, z_2, Z_1)P(z_2 \mid x_1, y_1, Z_1)P(y_1 \mid x_1, Z_1)$$

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$$\mathbb{E}[H_0^1(x_1)] = \mathbb{E}[f^{REG}(\mathbf{V};\eta_0)] = \sum_{z_1,z_2} P(y_2 \mid \mathbf{x}^{(2)}, y_1, z_2, z_1)P(z_2 \mid x_1, y_1, z_1)P(y_1 \mid x_1, z_1)P(z_1)$$

REG Estimand for mSBD - Proof

$$\mathbb{E}[f^{REG}(\mathbf{V};\mathbf{H}_0)] = \mathbb{E}[H_0^1(x_1)] = C(P)$$

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IPW Estimand for mSBD

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$f^{IPW}(\mathbf{V};\Pi) := W_0^n I_{\mathbf{y}}(\mathbf{Y})$ for

IPW Estimand for mSBD

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IPW Estimand for mSBD

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$$\mathbb{E}[W_0^2 I_{y_1,y_2}(Y_1, Y_2)] = \mathbb{E}\left[\frac{I_{x_1,x_2}(X_1, X_2)}{P(X_1 \mid Z_1)P(X_2 \mid \mathbf{Z}, X_1, Y_1)} I_{y_1,y_2}(Y_1, Y_2)\right]$$

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DR Estimand for mSBD

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$$f^{DR}(\mathbf{V}; \{\mathbf{H}_0, \Pi_0\}) = H_0^1(x_1) + \sum_{k=1}^{n} W_0^k \{H_0^{k+1}(x_{k+1}) - H_0^k(X_k)\},$$

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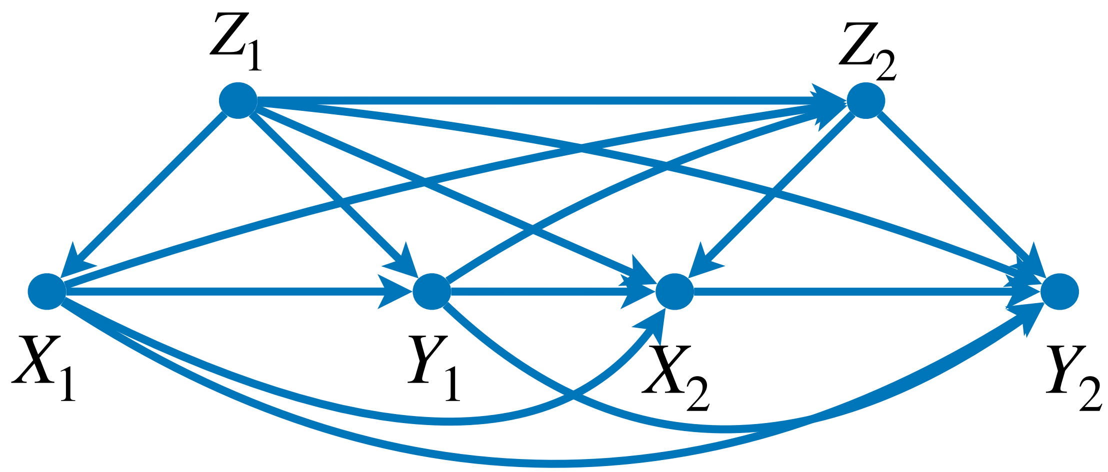
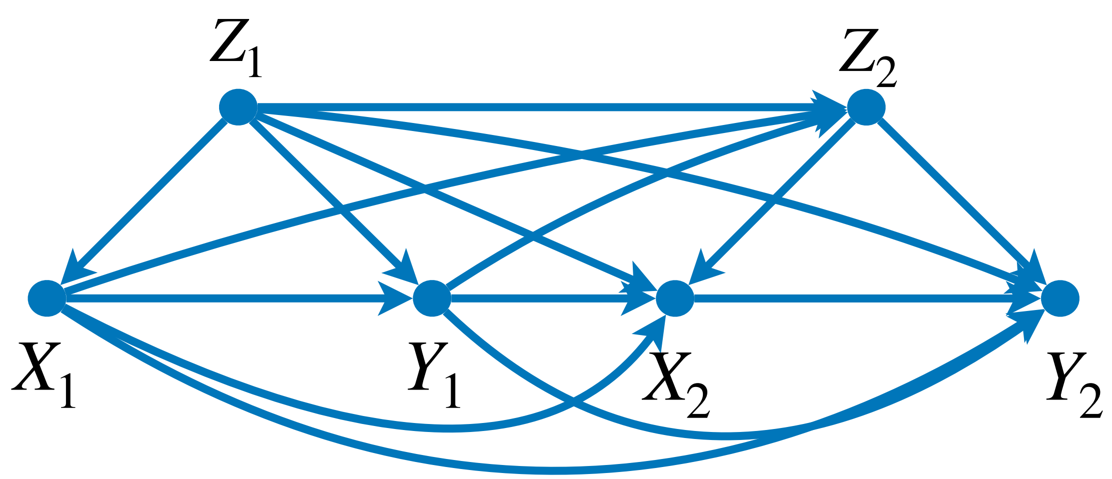
DR Estimand for mSBD — Example 1



$$f^{DR}(\mathbf{V}; \{\mathbf{H}_0, \mathbf{W}_0\}) = H_0^1(x_1) + W_0^1(H_0^2(x_2) - H_0^1(X_1)) + W_0^2(I_{\mathbf{y}}(\mathbf{Y}) - H_0^2(X_2))$$

DR Estimand for mSBD — Example 1



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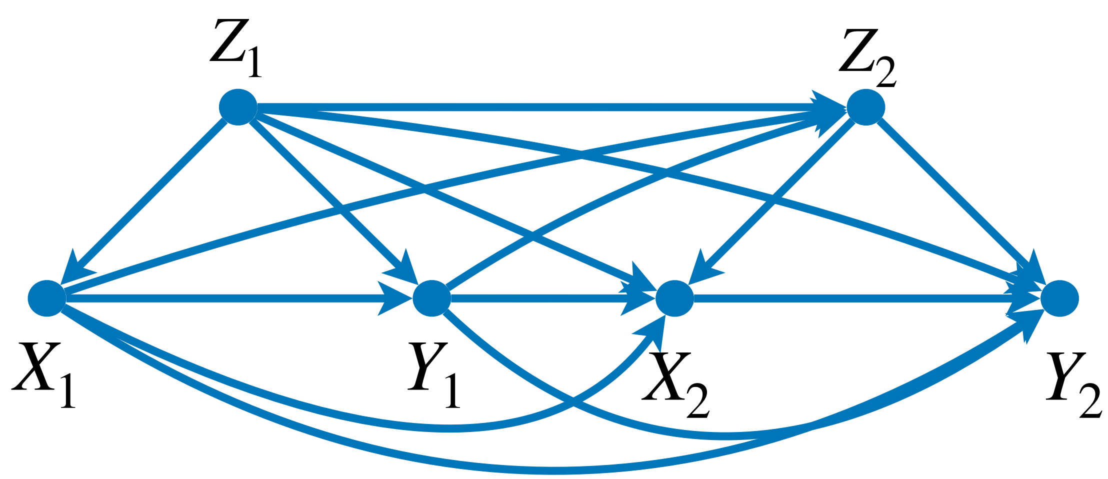
DR Estimand for mSBD — Example 1



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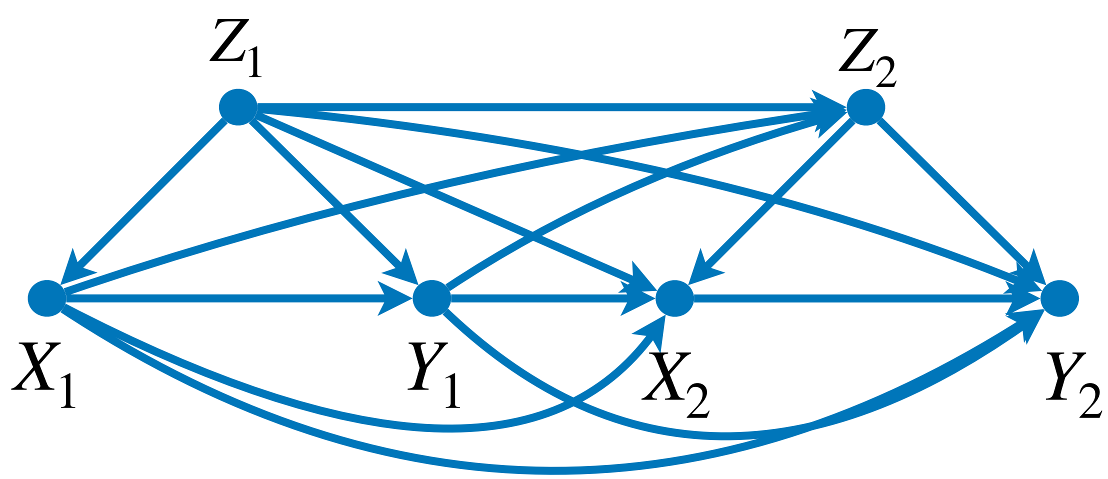
DR Estimand for mSBD — Example 1



$$f^{DR}(\mathbf{V}; \{\mathbf{H}_0, \mathbf{W}_0\}) = H_0^1(x_1) + W_0^1(H_0^2(x_2) - H_0^1(X_1)) + W_0^2(I_{\mathbf{y}}(\mathbf{Y}) - H_0^2(X_2))$$

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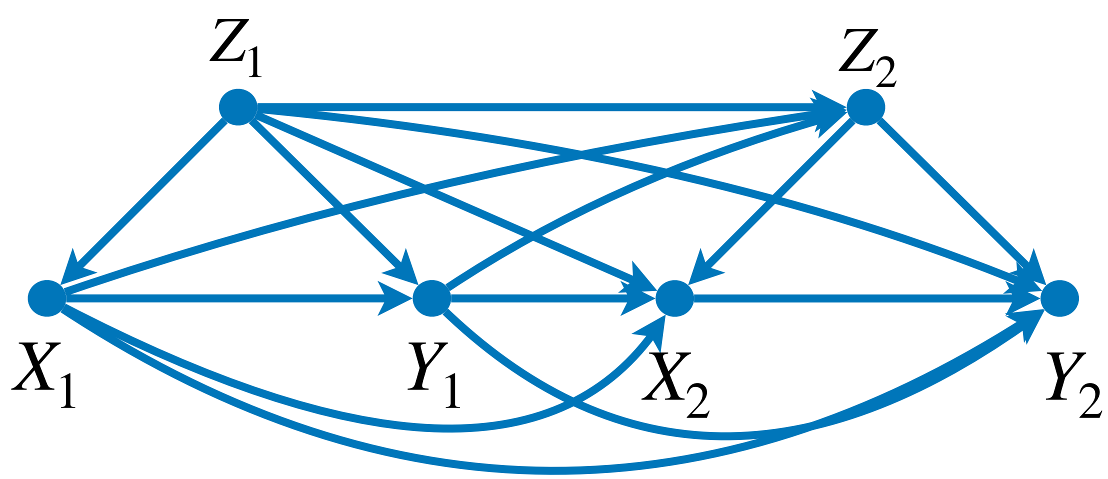
DR Estimand for mSBD — Example 1



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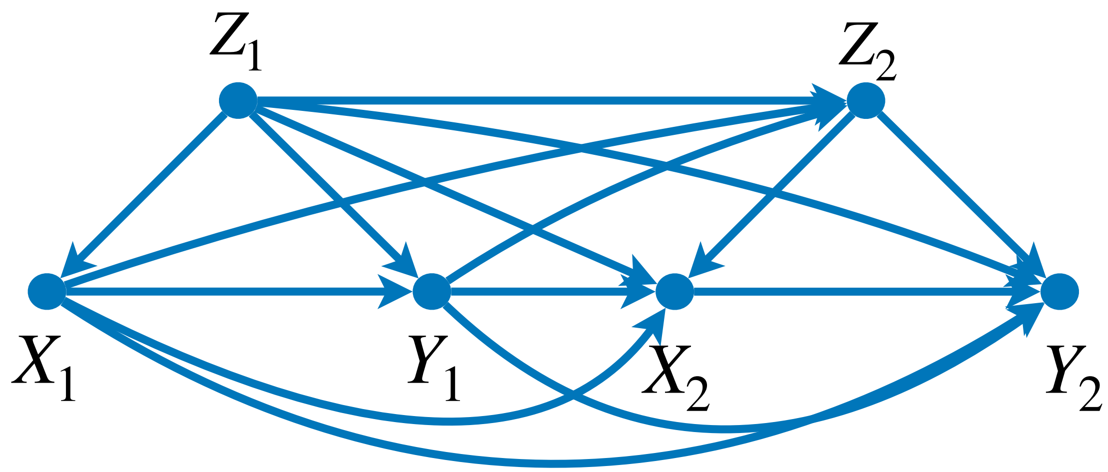
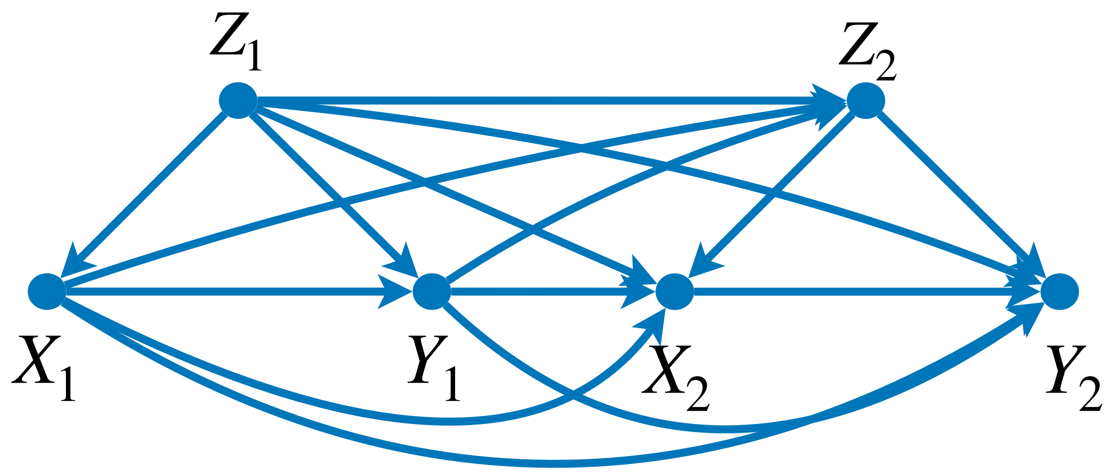
DR Estimand for mSBD — Example 2

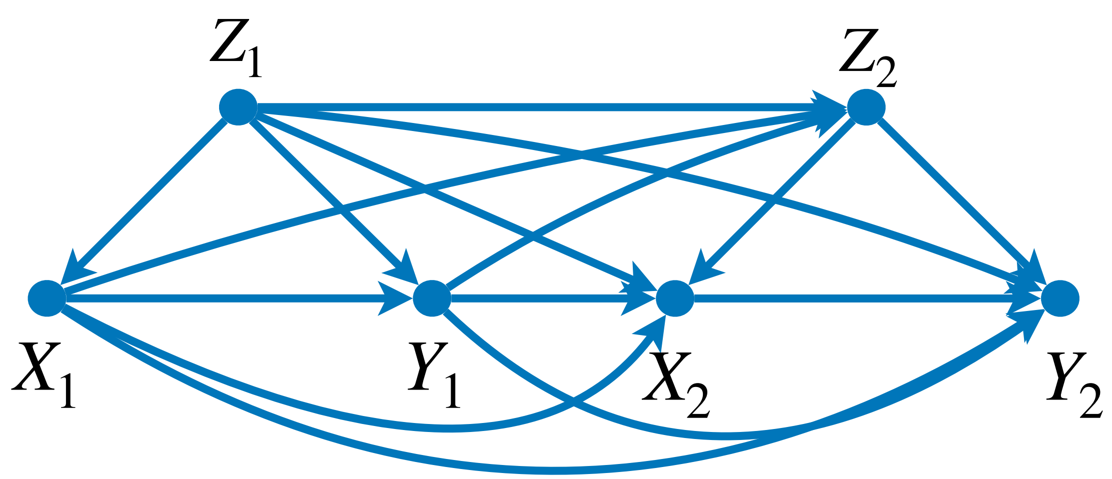


DR Estimand for mSBD — Example 2



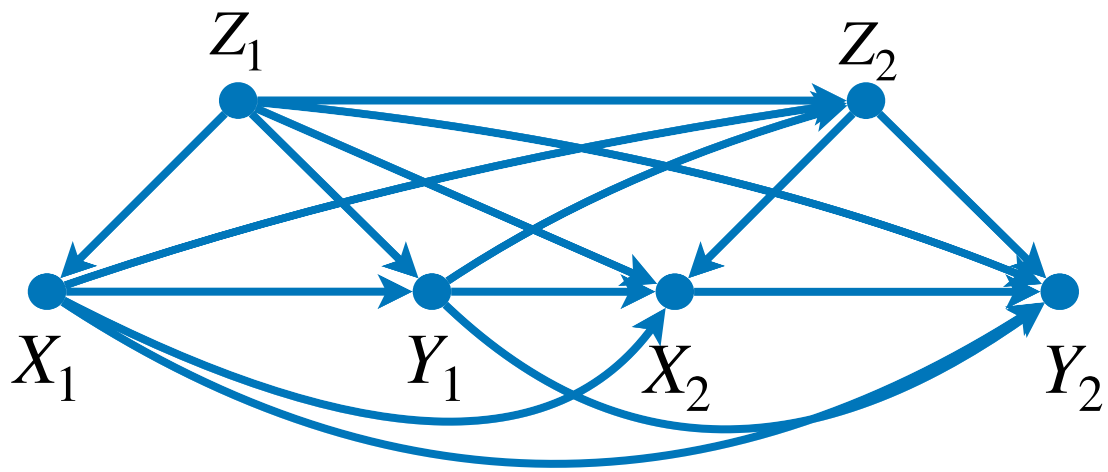
$$W_0^2 = \frac{I_{x_1}(X_1)}{\pi_0^1(Z_1)} \frac{I_{x_2}(X_2)}{\pi_0^2(X_1, \mathbf{Z}^{(2)}, Y_1)}$$

DR Estimand for mSBD — Example 2



$$W_0^2 = \frac{I_{x_1}(X_1)}{\pi_0^1(Z_1)} \frac{I_{x_2}(X_2)}{\pi_0^2(X_1, \mathbf{Z}^{(2)}, Y_1)} \qquad W_0^1 = \frac{I_{x_1}(X_1)}{\pi_0^1(Z_1)}$$

DR Estimand for mSBD — Example 2



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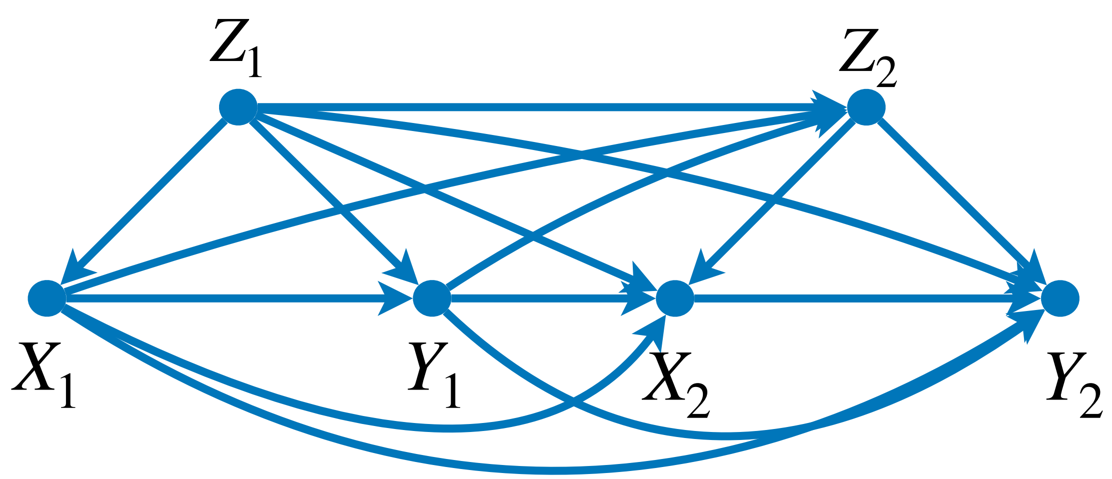
$$\pi_0^1(Z_1) = P(X_1 \mid Z_1)$$

DR Estimand for mSBD — Example 2



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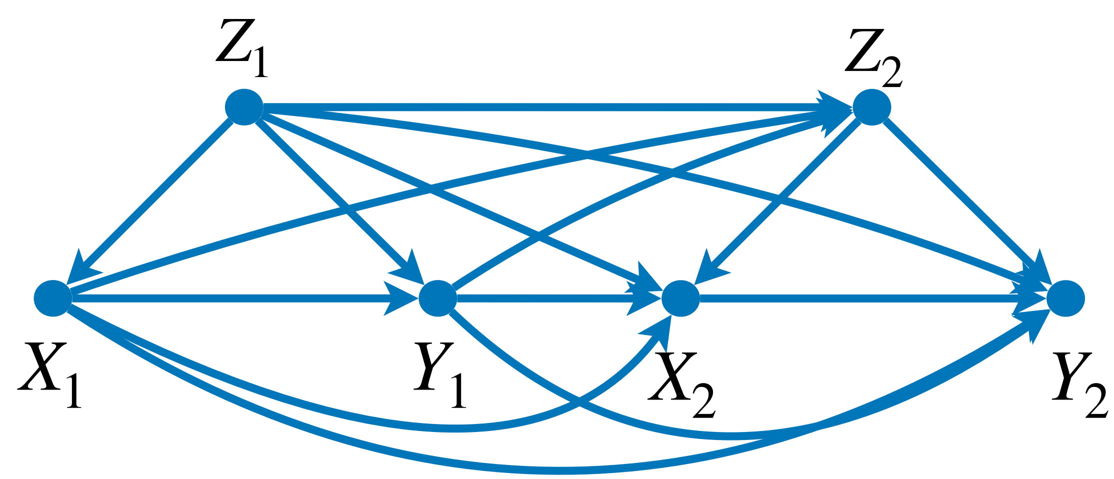
$$\pi_0^1(Z_1) = P(X_1 \mid Z_1) \qquad \pi_0^2(X_1, \mathbf{Z}^{(2)}, Y_1) = P(X_2 \mid X_1, \mathbf{Z}^{(2)}, Y_1)$$

DR Estimand for mSBD — Example 2



$$f^{DR}(\mathbf{V}; \{\mathbf{H}_0, \mathbf{W}_0\}) = H_0^1(x_1) + W_0^1(H_0^2(x_2) - H_0^1(X_1)) + W_0^2(I_{\mathbf{y}}(\mathbf{Y}) - H_0^2(X_2))$$

$$W_0^2 = \frac{I_{x_1}(X_1)}{\pi_0^1(Z_1)} \frac{I_{x_2}(X_2)}{\pi_0^2(X_1, \mathbf{Z}^{(2)}, Y_1)} \qquad W_0^1 = \frac{I_{x_1}(X_1)}{\pi_0^1(Z_1)}$$

$$\pi_0^1(Z_1) = P(X_1 \mid Z_1) \qquad \pi_0^2(X_1, \mathbf{Z}^{(2)}, Y_1) = P(X_2 \mid X_1, \mathbf{Z}^{(2)}, Y_1)$$

Orthogonality of the DR estimand - Proof Sketch 1

$f^{DR}(\mathbf{V}; \{\mathbf{H}, \Pi\}) = H^1(x_1) + \sum_{i=1}^{n} W^i\{H^{i+1}(x_{i+1}) - H^i(X_i)\}$ is an orthogonal estimand.

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Debiasedness &
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A proof is omitted due to its complexness. Check [Rotnitzky, Robins, Babino, 2017] for the detailed proof.

Debiasedness & Doubly Robustness

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1. **Debiasedness**: $\mathbb{E}[f^{DR}(\mathbf{V}; \{\mathbf{H}, \Pi\})]$ converges to the mSBD adjustment $C(P)$ at $N^{-1/2}$ rate if H^i, π^i converge to H_0^i, π_0^i at $N^{-1/4}$ rate.

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2. **Doubly robustness**: $\mathbb{E}[f^{DR}(\mathbf{V}; \{\mathbf{H}, \Pi\})] = C(P)$ if either $H^i = H_0^i$ or $\pi^i = \pi_0^i$ for all $i \in \{1, 2, \cdots, n\}$.

Finite samples

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2 (first term) 1 (second term)

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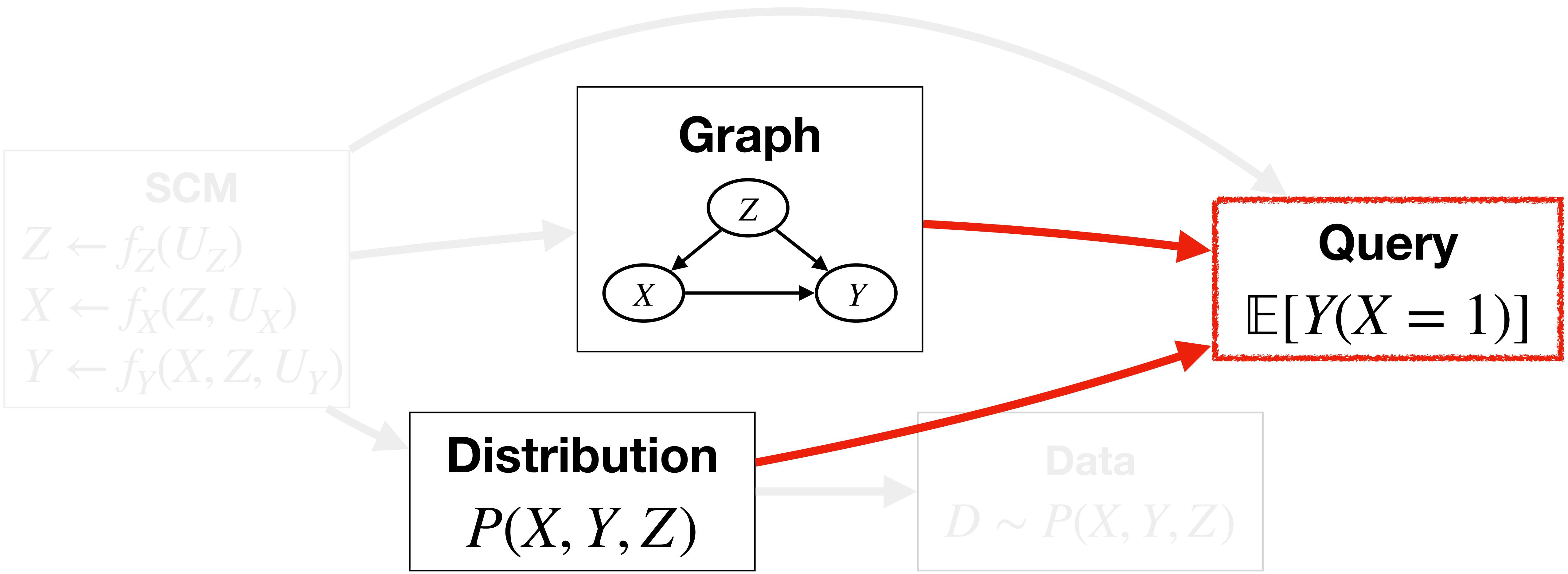
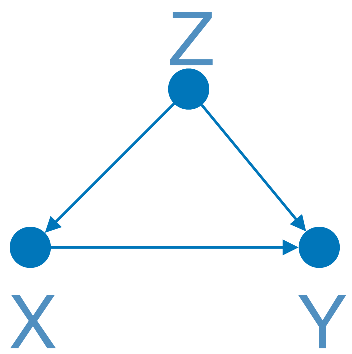
General case — Causal functional represented as a function of mSBDs

Algorithmic approach to Identifiability

Recap: Causal Effect Identification: Big Picture (1)



Recap: Causal Effect Identification



Causal graph (G)

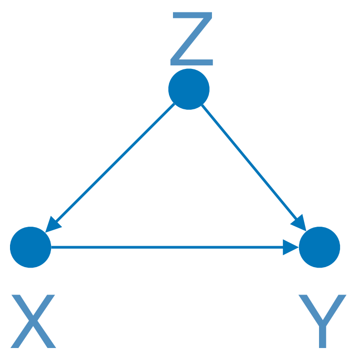
$P(Z, X, Y)$

Distribution on G (P)

$$Q_0 := \mathbb{E}[Y \mid do(x)]$$

Causal Query (Q_0)

Recap: Causal Effect Identification



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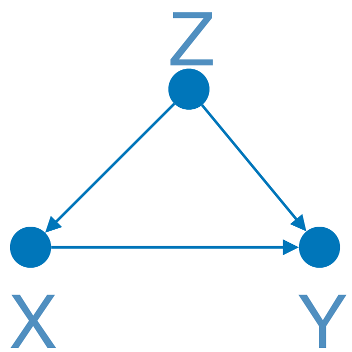

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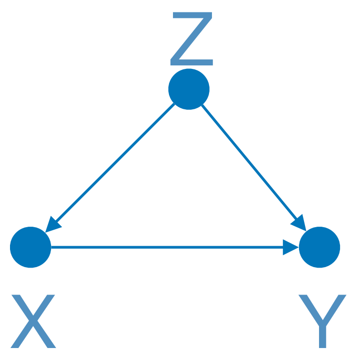

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Given {G, P, Q_0},

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finds the functional $C(P)$ s.t. $C(P) = Q_0$.

ID(G, P, Q_0)

ID algorithm

$C(P) = \sum_z \mathbb{E}[Y \mid x, z]P(z)$

Causal functional

$C(P)$ **s.t.** $C(P) = Q_0$

Conditional Independence and d-Separation

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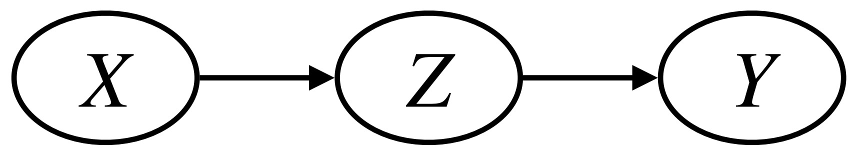

D-separation: The path from X to Y is blocked by Z;

$$(X \perp\!\!\!\perp Y \mid Z)_G$$

Conditional independence: Then, in the corresponding distribution P induced by the SCM,

$$(X \perp\!\!\!\perp Y \mid Z)_P\text{; i.e., } P(Y \mid X, Z) = P(Y \mid Z).$$

Intervention and Causal Graph $G_{\overline{X}}$

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$G_{\overline{X}}$: Intervention $do(X=1)$ (replacing $X \leftarrow f_X$ to $X \leftarrow x$) induces the graph where the incoming edges to X is cut.

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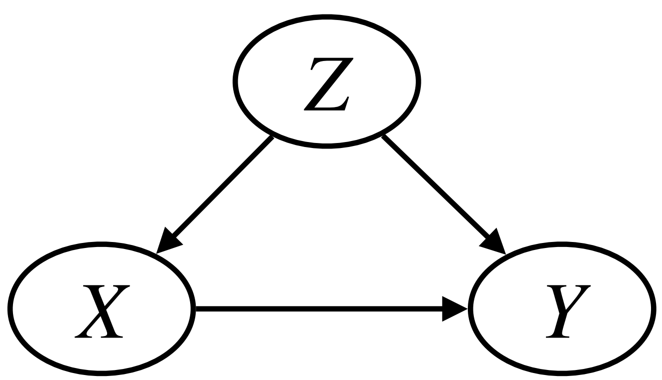

SCM

$$Z \leftarrow f_Z(U_Z)$$
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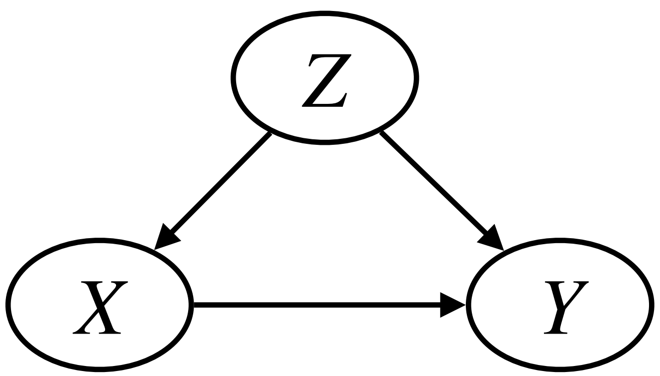

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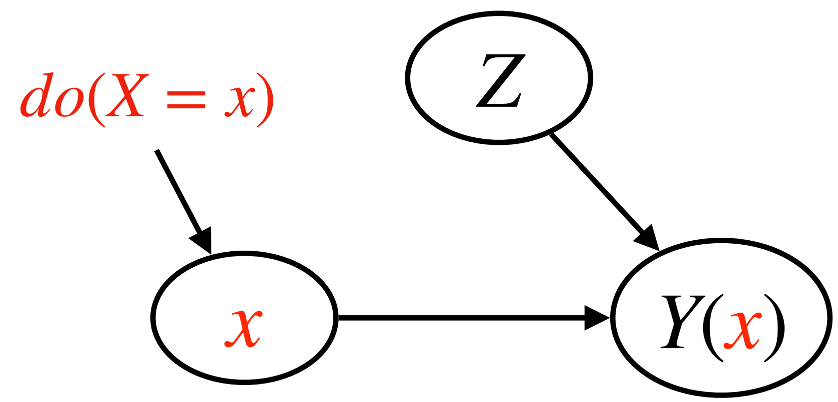

SCM ($do(X=1)$)

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$$X \leftarrow x$$
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Counterfactual and Causal Graph $G_{\underline{X}}$

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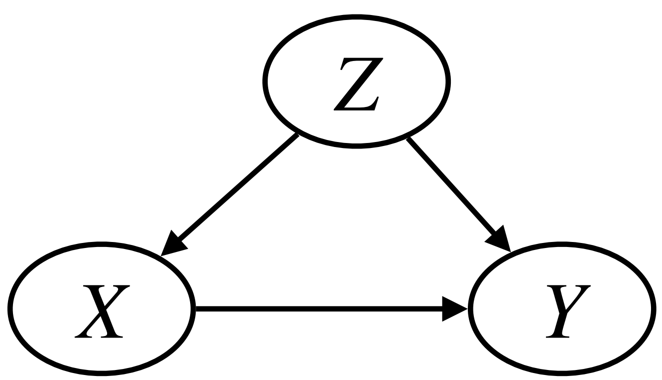

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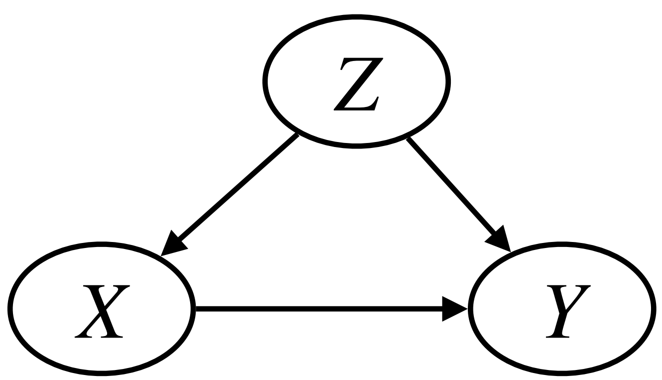

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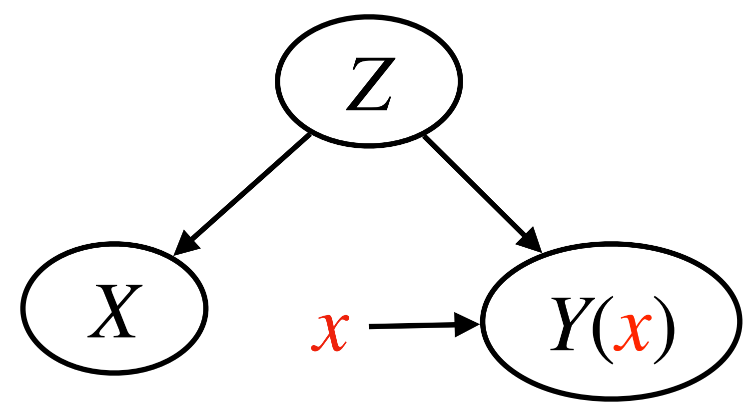

SCM'

$$Z \leftarrow f_Z(U_Z)$$
$$X \leftarrow f_X(U_X, Z)$$
$$Y(x) \leftarrow f_Y(Z, X = 1, U_Y)$$

do-Calculus - R1: Excluding Observation

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Rule 1: If $(Y \perp\!\!\!\perp Z \,|\, X, W)_{G_{\overline{X}}}$, then

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$G_{\overline{X}}$ is a graph for $P(\,\cdot\,|\, do(x))$.

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do-Calculus - R2: Exchanging Intervention/Observation

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do-Calculus - R3: Excluding Intervention

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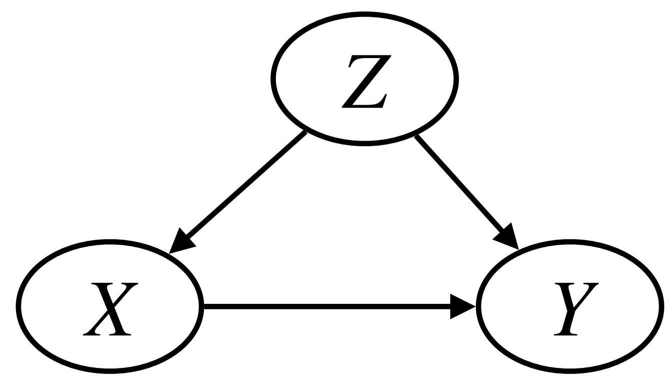
Rule 3: If $(Y \perp\!\!\!\perp Z \mid X, W)_{G_{\overline{X}\overline{Z(W)}}}$, where $Z(W) := Z \backslash An(W)_{G_{\overline{X}}}$

$$P(y \mid do(x), do(z), w) = P(y \mid do(x), w)$$

If (Y, Z) are conditionally independent after intervening $\{X, Z\}$, then Z is a redundant intervention, so it can be removed.

do-Calculus: Example to Back-door



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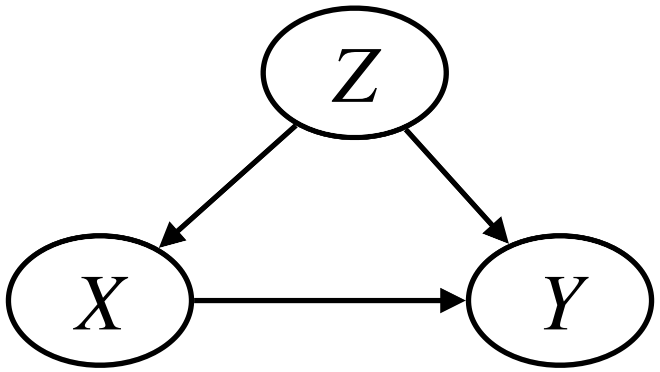

$$\mathbb{E}[Y \mid do(x)] = \sum_z \mathbb{E}[Y \mid do(x), z] P(z \mid do(x))$$ Marginalization over $Z = z$

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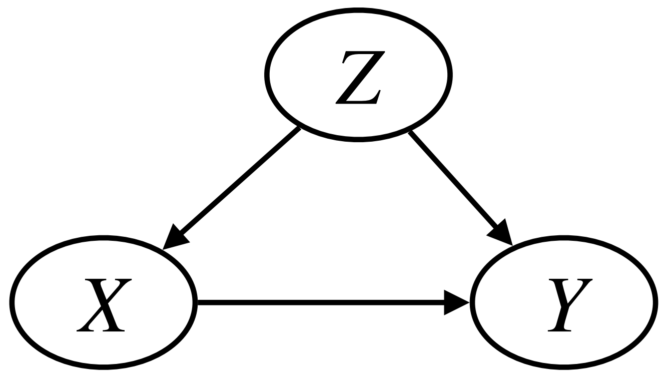

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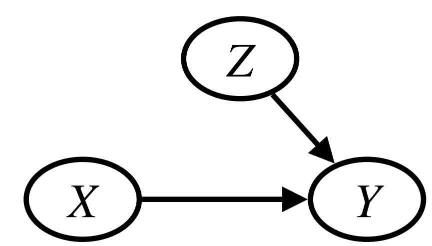

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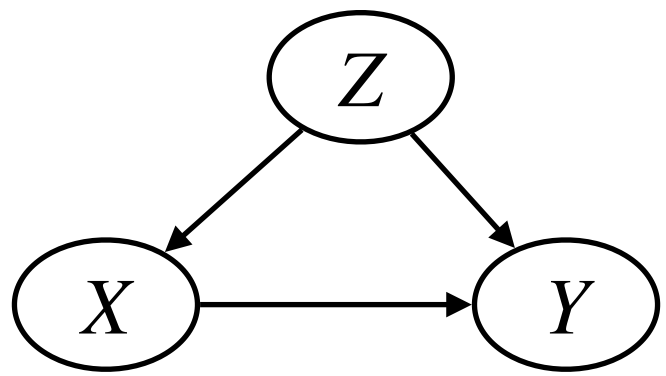

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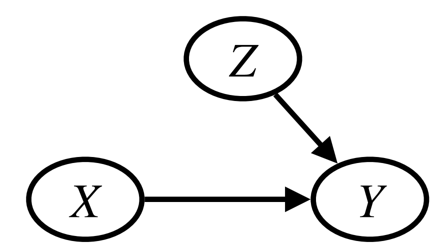

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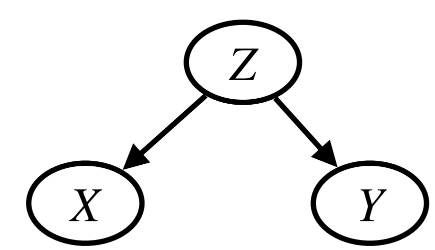

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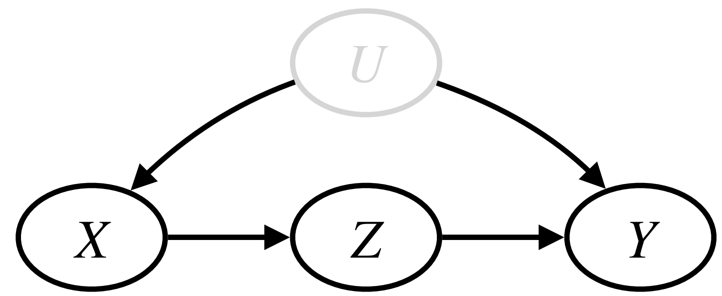

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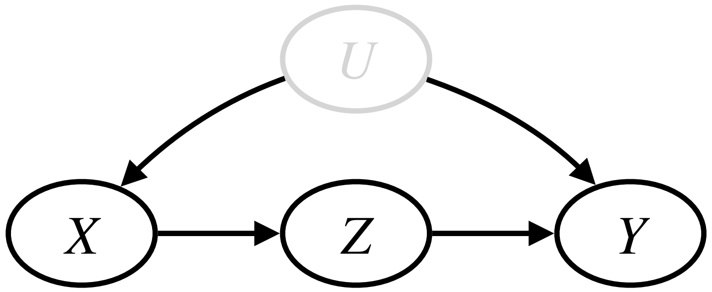

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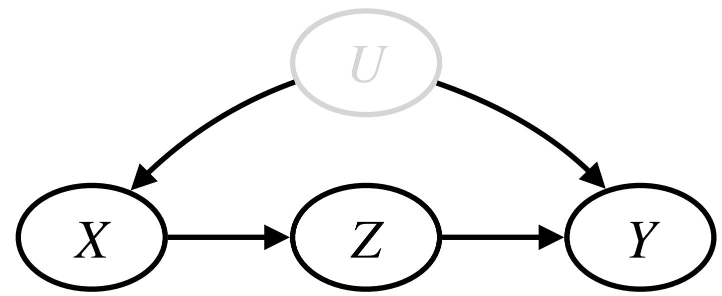

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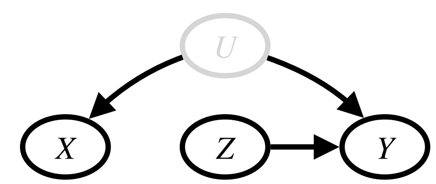

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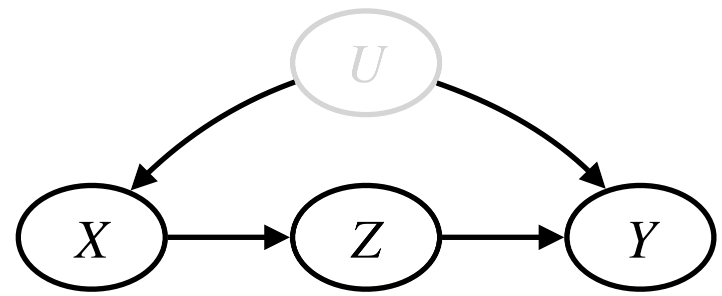

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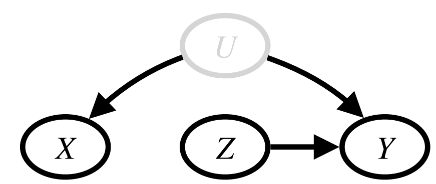

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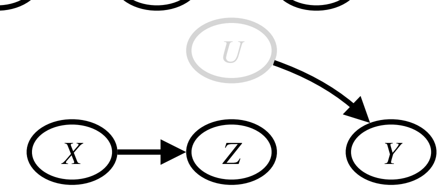

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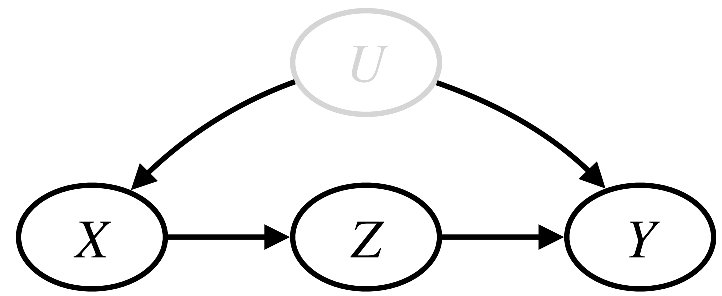

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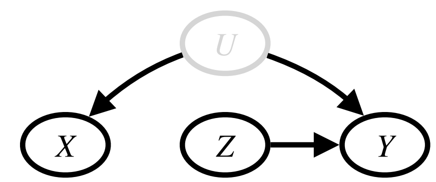

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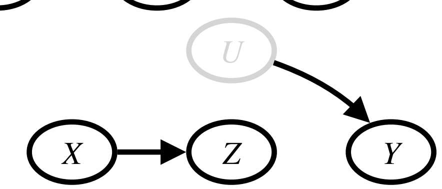

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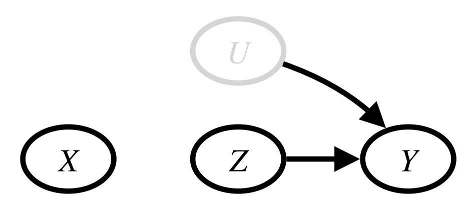

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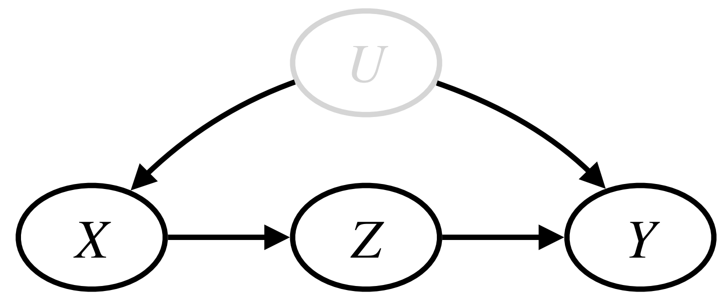

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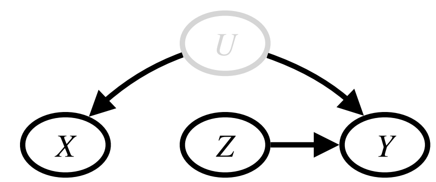

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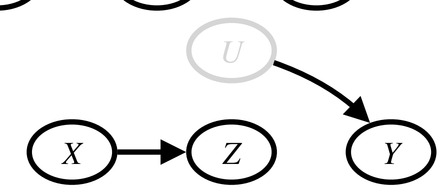

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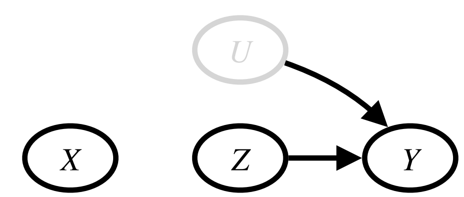

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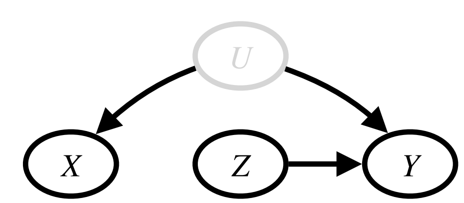

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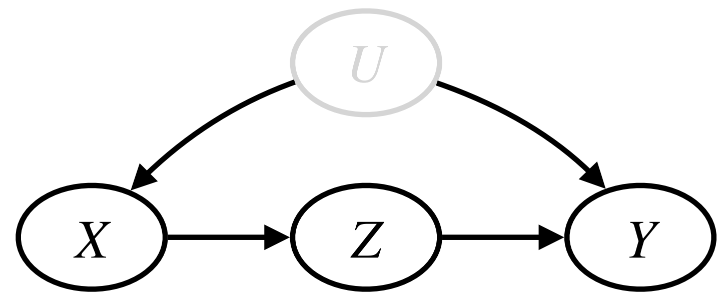

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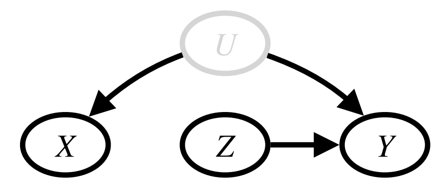

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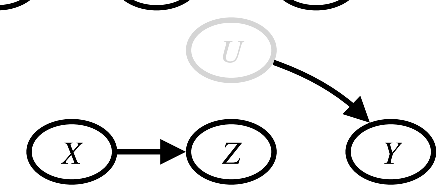

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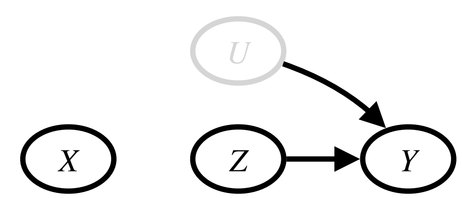

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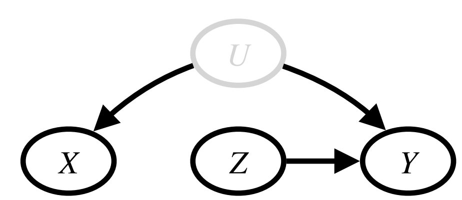

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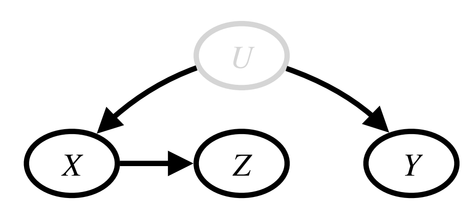

do-Calculus:
Soundness and Completeness

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The causal effect is identifiable (i.e., $\mathbb{E}[Y | do(x)]$ can be written as a function $C(P)$) if-and-only-if there is a series of do-calculus application.

do-Calculus: Algorithmic Procedure

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There is an algorithm (e.g., CausalFusion) that finds such rules for identifying causality.

Soundness and Completeness of DML-ID

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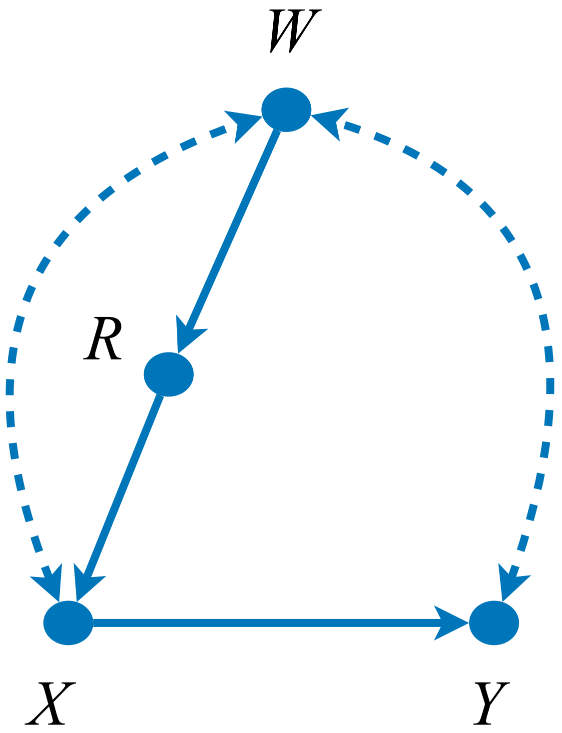
Example of DML-ID: Napkin



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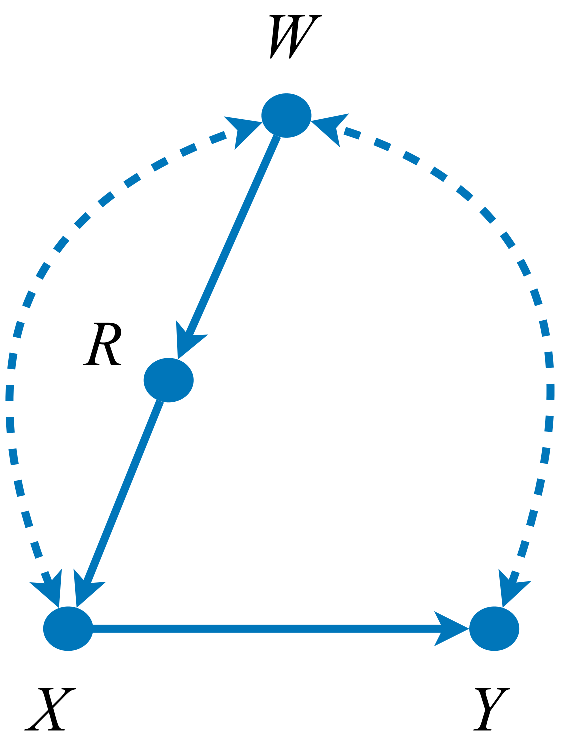

$$P(y \mid do(x)) = \frac{\sum_w P(x, y \mid r, w) P(w)}{\sum_w P(x \mid r, w) P(w)} = \frac{M^b}{M^a}$$

Construction of DML estimator

What we learned so far

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mSBD adjustment:

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Orthogonal (and Doubly-Robust) estimand for mSBD:

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Deriving an Orthogonal Estimand - 1

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For an mSBD operator M, let $\mu_{M_0} := \mathbb{E}[f^{DR}(V; \{\mathbf{H}_0, \Pi_0\})]$ denote an expectation of the DR (mSBD) estimand with true nuisances $\mathbf{H}_0, \Pi_0$.

Deriving an Orthogonal Estimand - 2

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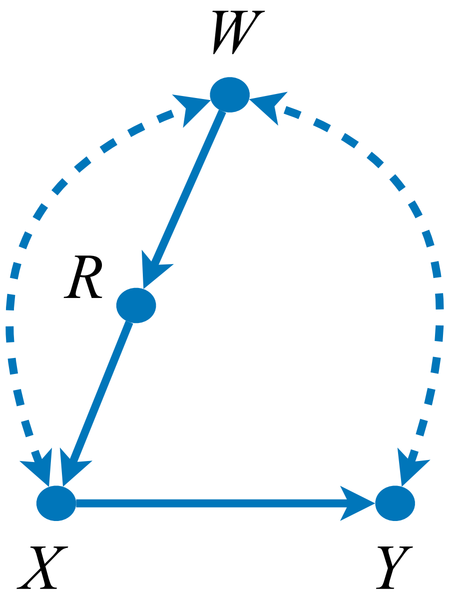
Example for deriving an Orthogonal Estimand - 1



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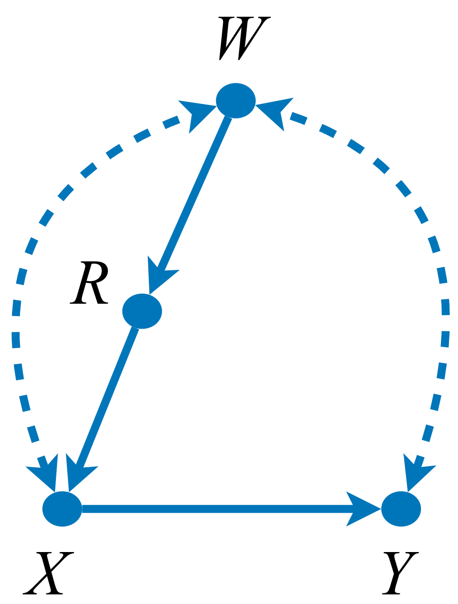

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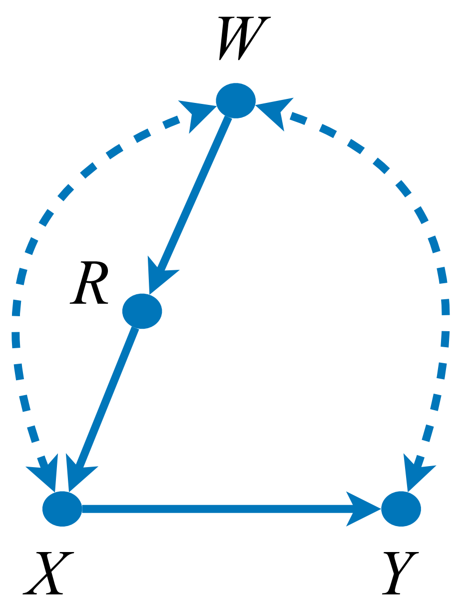

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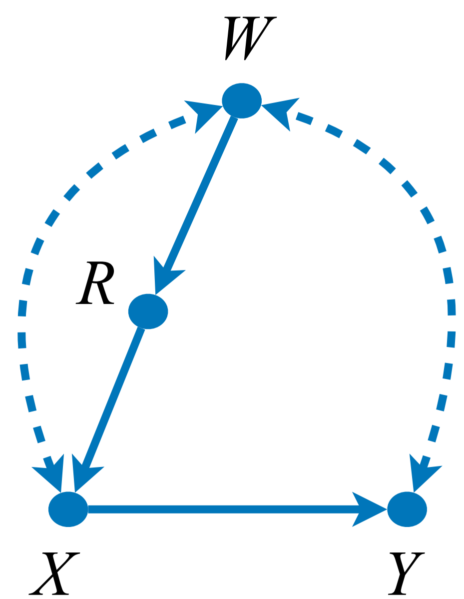

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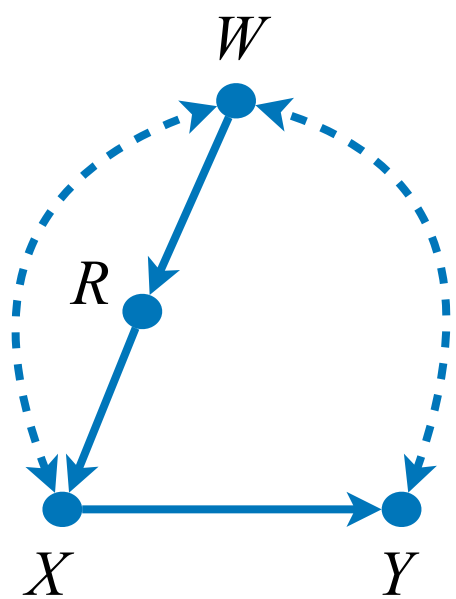

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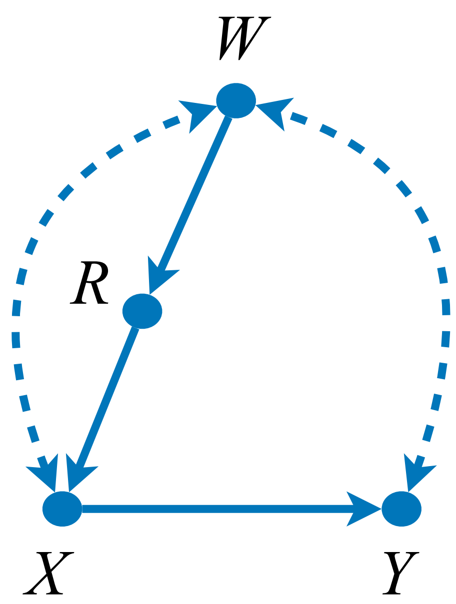

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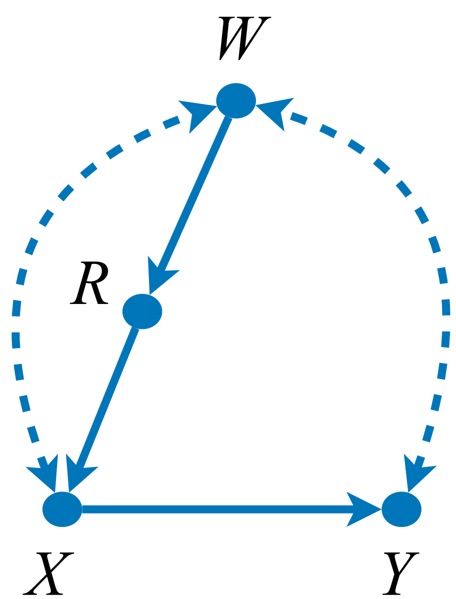

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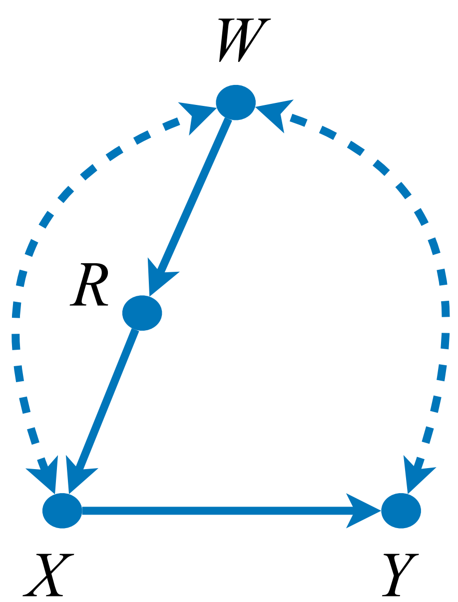

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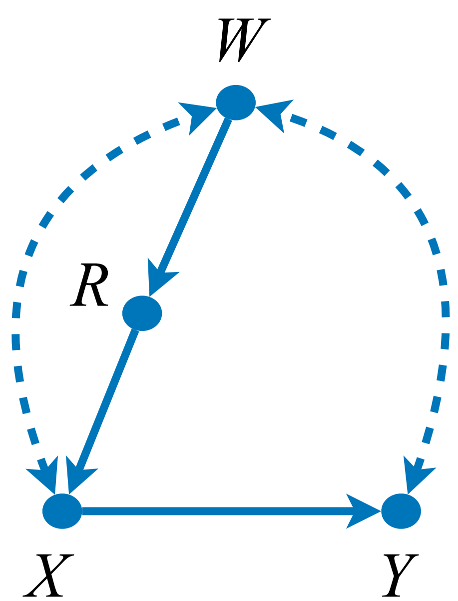

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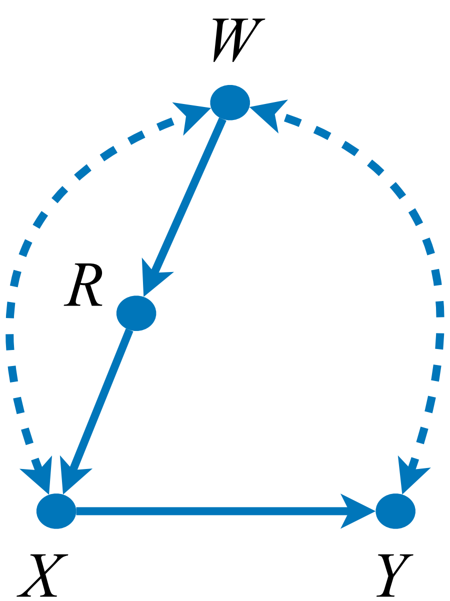

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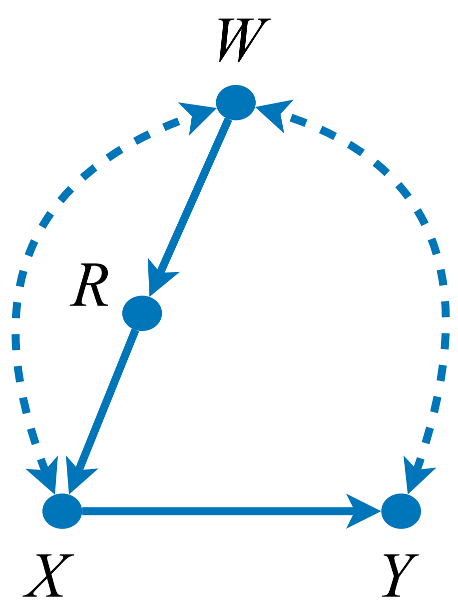

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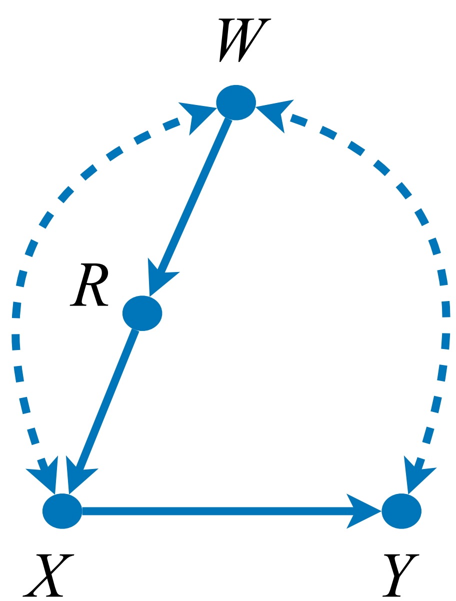

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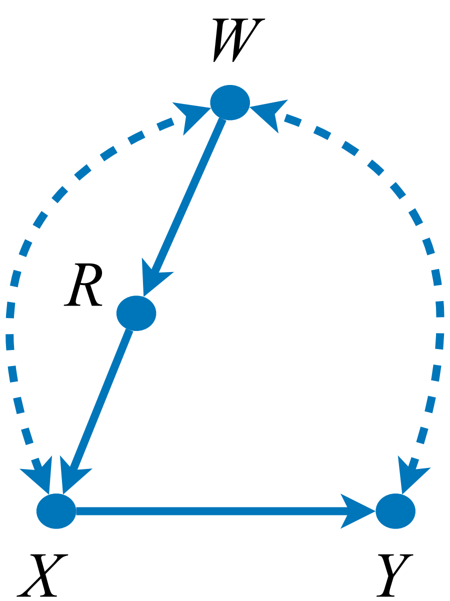

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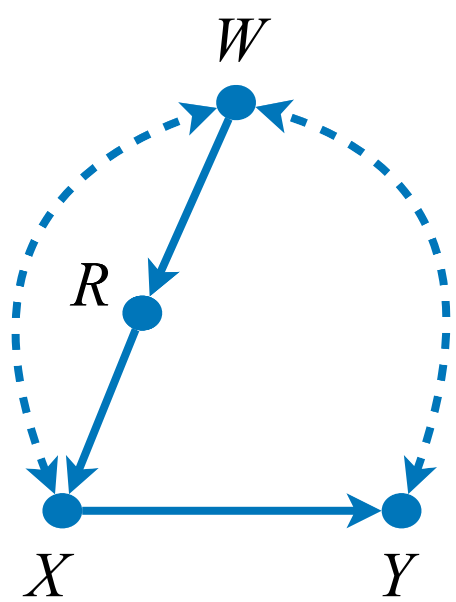

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Example for deriving an Orthogonal Estimand - 3

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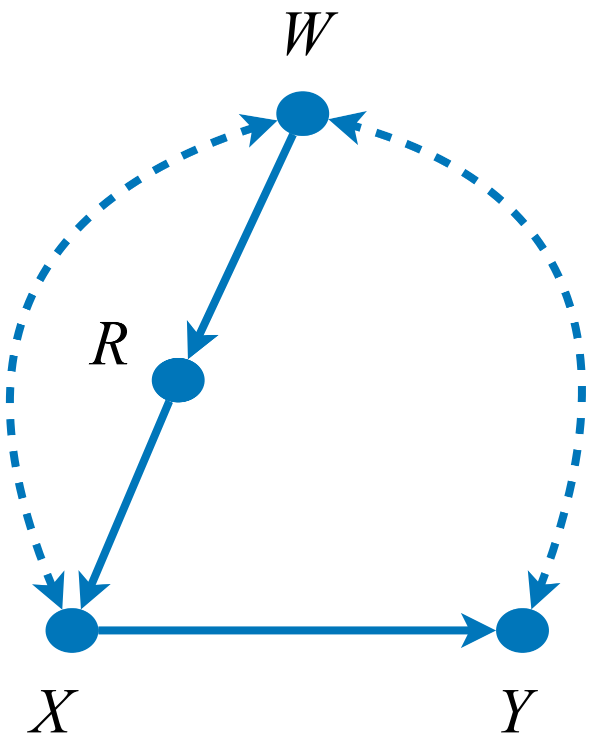
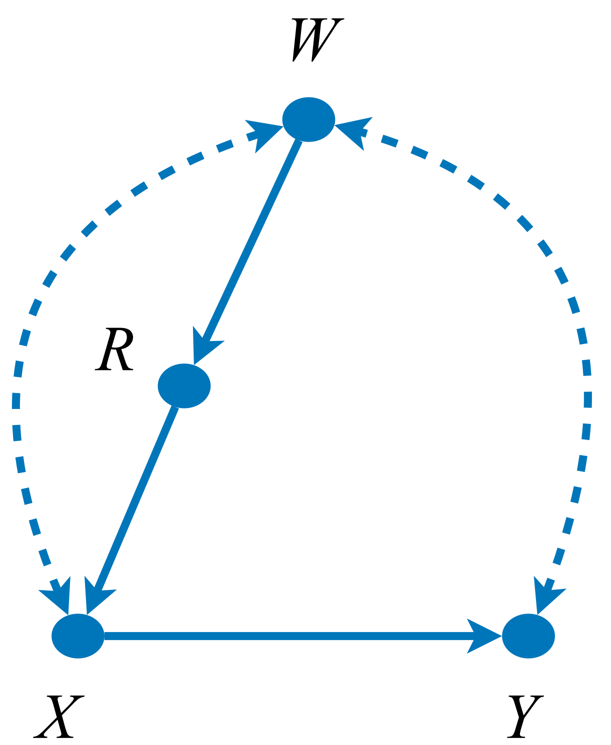


Example for deriving an Orthogonal Estimand - 3



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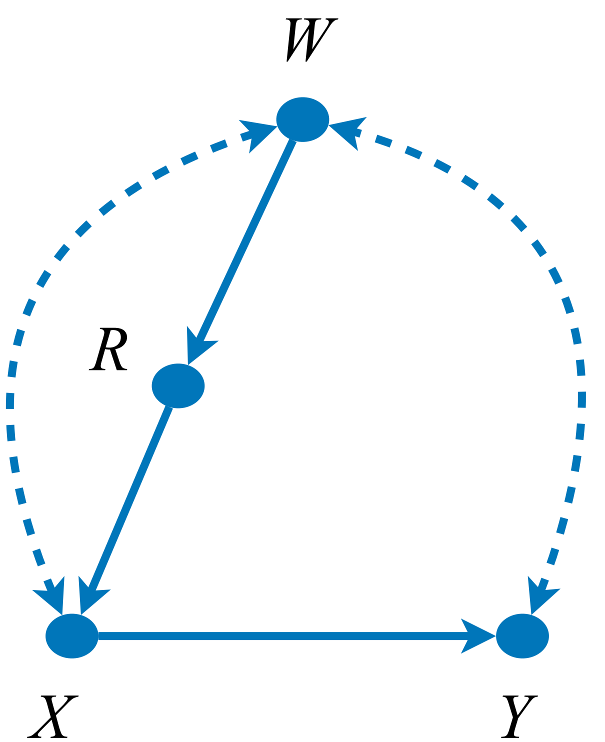
Example for deriving an Orthogonal Estimand - 3



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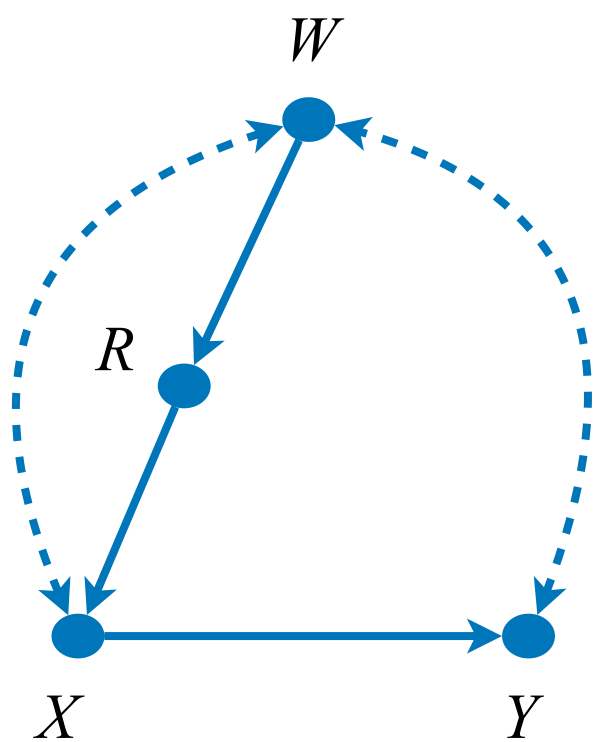
Example for deriving an Orthogonal Estimand - 3



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Constructing DML estimators

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Given

$$P_{\mathbf{x}}(\mathbf{y}) = A(\{\mu_{M_0^a}\}_{a=1}^{m}),$$

the DML estimator is given as follow:

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$$T = A(\{\hat{\mu}_{M^a}\}_{a=1}^{m}),$$

for $\hat{\mu}_{M^a} := \mathbb{E}_D\left[f^{DR}(\mathbf{V}; \{\hat{\mathbf{H}}, \hat{\mathbf{W}}\})\right]$ s.t. $\hat{\eta}$ are trained using samples independent to D

Error Analysis - 1

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$$T - C(P) = O_P(N^{-1/2}) + O_P\left(\sum_{a=1}^{m} \epsilon(\hat{\mu}_{M^a}) \right)$$

$$\text{for } \epsilon\left(\hat{\mu}_{M^a}\right) := \sum_{i=1}^{n} \| H^i - H_0^i \| \| \pi^i - \pi_0^i \|,$$

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1. **Debiasedness**: T converges to $C(P)$ at $N^{-1/2}$ rate if all nuisances for μ_{M^a} (i.e., $\{H^i, \pi^i\}_{i=1}^n$) converge at $N^{-1/4}$ rate.

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Error Analysis - Proof 1

$$T - P_{\mathbf{x}}(\mathbf{y}) = O_P\left(\sum_{a=1}^{m} \{\hat{\mu}_{M^a} - \mu_{M_0^a}\}\right)$$

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$$T - P_{\mathbf{x}}(\mathbf{y}) = O_P\left(\sum_{a=1}^{m} \{\hat{\mu}_{M^a} - \mu_{M_0^a}\} \right)$$

Note that

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$$T - P_{\mathbf{x}}(\mathbf{y}) = O_P\left(\sum_{a=1}^{m}\{\hat{\mu}_{M^a} - \mu_{M_0^a}\}\right)$$

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DML estimator for the Napkin Graph

$$P_x(y) = \frac{\sum_w P(x, y \mid r, w)P(w)}{\sum_w P(x \mid r, w)P(w)} = \frac{\mu_{M_0^a}}{\mu_{M_0^b}}$$

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where samples for training nuisances and evaluating f^a, f^b are independent (sample-splitting).

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Then, the DML estimator for the example is

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Error Analysis - Napkin Example 1

$$T - P_x(y) = O\left(\sum_{i \in \{a,b\}} \hat{\mu}_{M^i} - \mu_{M_0^i}\right)$$

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$$\begin{aligned}
T - P_x(y) &= \frac{\hat{\mu}_{M^a}}{\hat{\mu}_{M^b}} - \frac{\mu_{M_0^a}}{\mu_{M_0^b}} \\
&= \frac{\hat{\mu}_{M^a}}{\hat{\mu}_{M^b}} - \frac{\hat{\mu}_{M^a}}{\mu_{M_0^b}} + \frac{\hat{\mu}_{M^a}}{\mu_{M_0^b}} - \frac{\mu_{M_0^a}}{\mu_{M_0^b}} \\
&= \hat{\mu}_{M^a}\left(\frac{1}{\hat{\mu}_{M^b}} - \frac{1}{\mu_{M_0^b}} \right) + \frac{1}{\mu_{M_0^b}}(\hat{\mu}_{M^a} - \mu_{M_0^a}) \\
&= O(\sum_{i \in \{a,b\}} \hat{\mu}_{M^i} - \mu_{M_0^i})
\end{aligned}$$

Error Analysis - Napkin Example 2

$$T - P_x(y) = O\left(\sum_{i \in \{a,b\}} \hat{\mu}_{M^i} - \mu_{M_0^i} \right)$$

$$T := \frac{\hat{\mu}_{M^a}}{\hat{\mu}_{M^b}}$$

Error Analysis - Napkin Example 2

$$T - P_x(y) = O\left(\sum_{i \in \{a,b\}} \hat{\mu}_{M^i} - \mu_{M_0^i}\right)$$

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Recall

$$\hat{\mu}_{M^a} - \mu_{M_0^a} = O_P(N^{-1/2}) + O_P\left(\| H^a - H_0^a \| \| \pi - \pi_0 \|\right)$$

$$\hat{\mu}_{M^b} - \mu_{M_0^b} = O_P(N^{-1/2}) + O_P\left(\| H^b - H_0^b \| \| \pi - \pi_0 \|\right),$$

where $\{H^i, \pi^i\}$ are nuisances for $M^i \in \{M^a, M^b\}$

Error Analysis - Napkin Example 2

$$T - P_x(y) = O\left(\sum_{i \in \{a,b\}} \hat{\mu}_{M^i} - \mu_{M_0^i} \right)$$

$$T := \frac{\hat{\mu}_{M^a}}{\hat{\mu}_{M^b}}$$

Recall

$$\hat{\mu}_{M^a} - \mu_{M_0^a} = O_P(N^{-1/2}) + O_P\left(\| H^a - H_0^a \| \| \pi - \pi_0 \|\right)$$

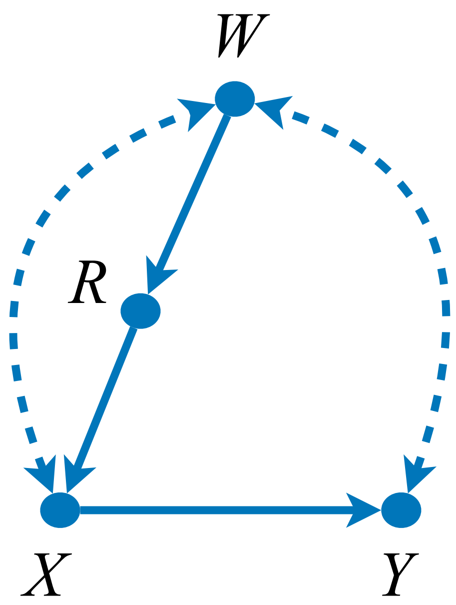
$$\hat{\mu}_{M^b} - \mu_{M_0^b} = O_P(N^{-1/2}) + O_P\left(\| H^b - H_0^b \| \| \pi - \pi_0 \|\right),$$

where $\{H^i, \pi^i\}$ are nuisances for $M^i \in \{M^a, M^b\}$

1. **Debiasedness**: T converges to $C(P)$ at $N^{-1/2}$ rate if H^a, H^b, π converge to H_0^a, H_0^b, π_0^i at $N^{-1/4}$ rate.
2. **Doubly robustness**: $T = P_x(y)$ if $H^a = H_0^a$ or $\pi^i = \pi_0^i$; and $H^b = H_0^b$ or $\pi^i = \pi_0^i$

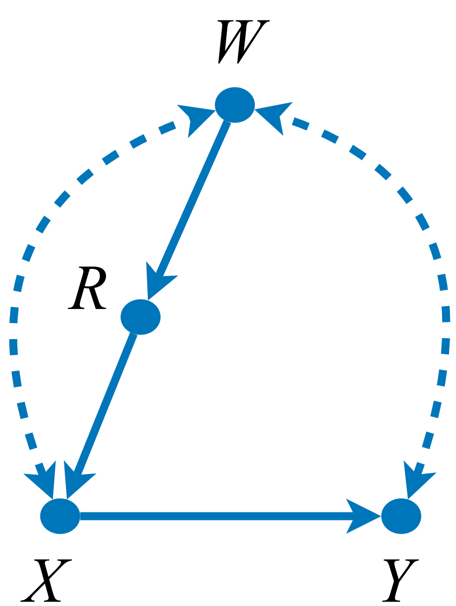
Empirical Evidence

Empirical result — Expected results



$$P_x(y) = C(P) = \frac{\sum_w P(x, y \mid r, w)P(w)}{\sum_w P(x \mid r, w)P(w)}$$

Empirical result — Expected results



$$P_x(y) = C(P) = \frac{\sum_w P(x, y \mid r, w) P(w)}{\sum_w P(x \mid r, w) P(w)}$$

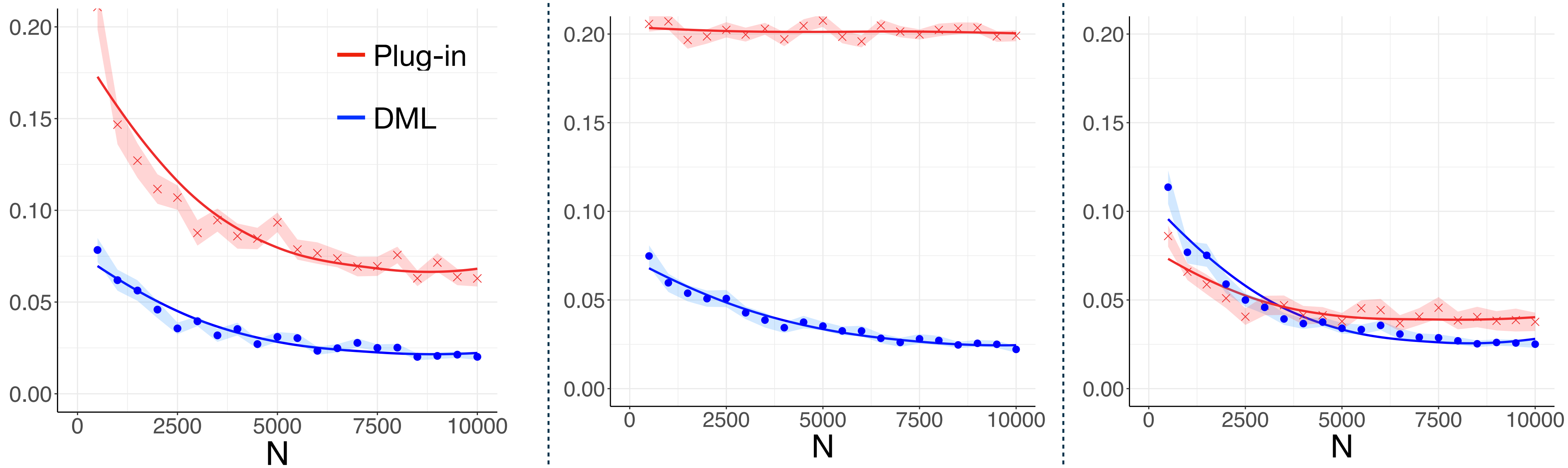
$$C(\hat{P}) = \frac{\sum_w \hat{P}(x, y \mid r, w) \hat{P}(w)}{\sum_w \hat{P}(x \mid r, w) \hat{P}(w)}$$

$C(\hat{P})$: the plug-in estimator, only viable estimator working for identifiable causal functional.

Empirical results

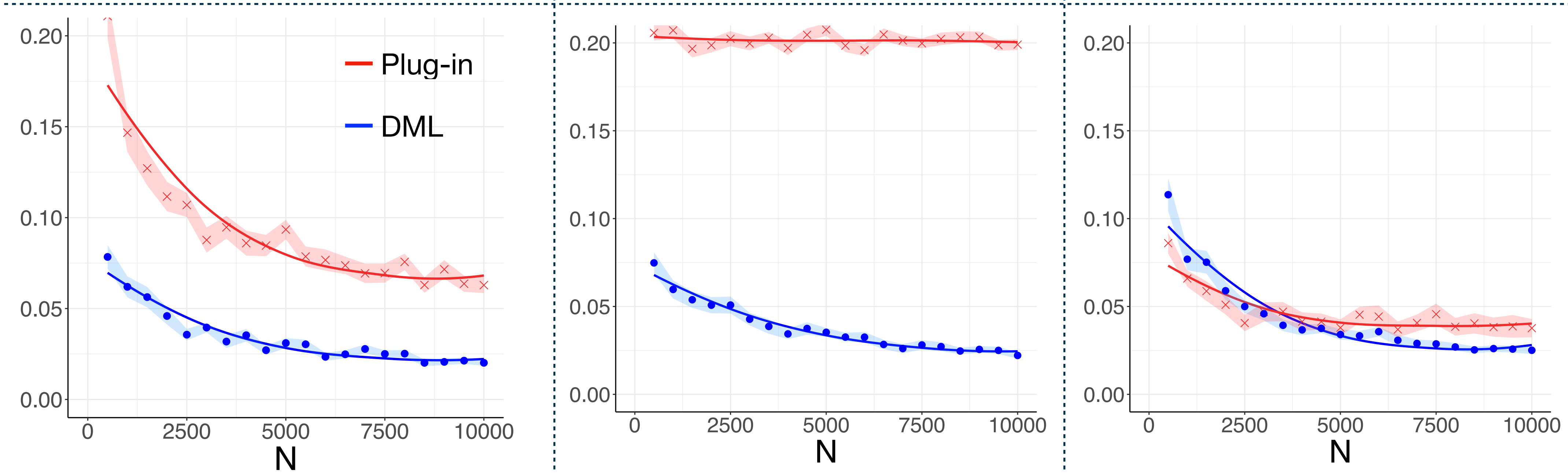
Empirical results

Both converge at rate $N^{-1/4}$	$P(x, y \mid r, w)$ misspecified.	$P(r \mid w)$ misspecified.



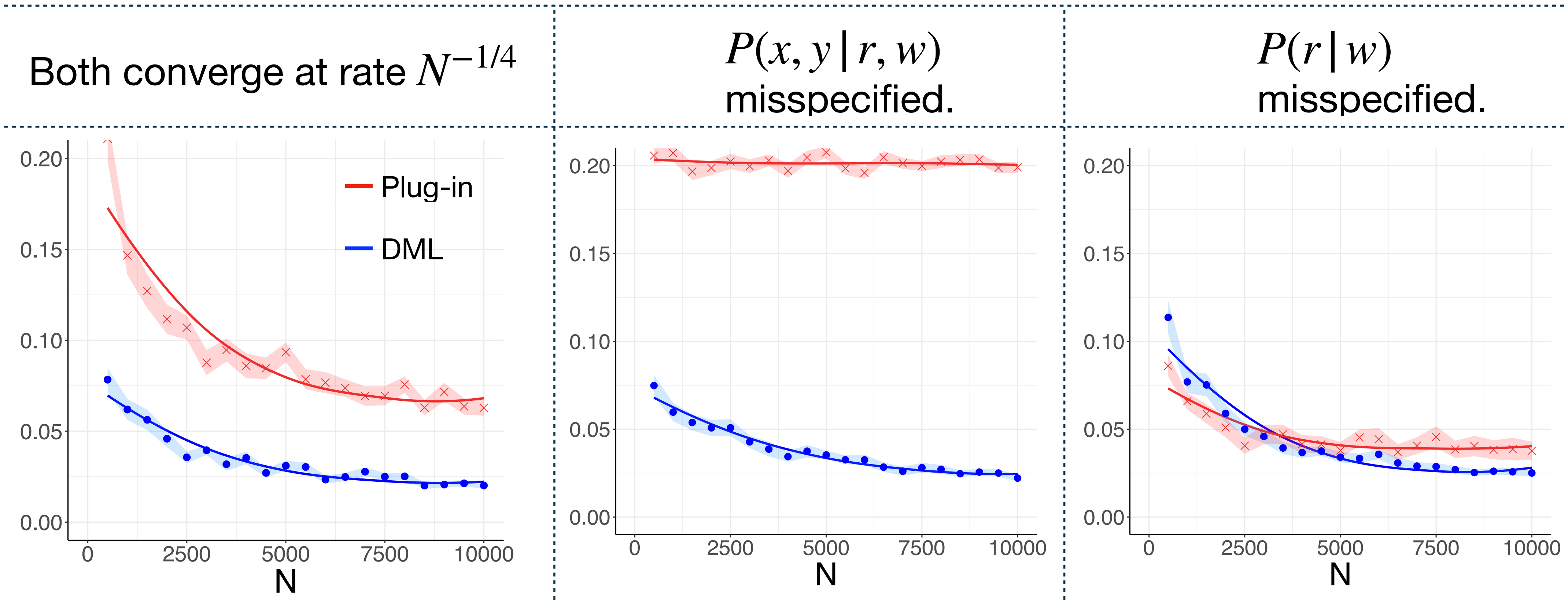
Empirical results

Both converge at rate $N^{-1/4}$	$P(x, y \mid r, w)$ misspecified.	$P(r \mid w)$ misspecified.



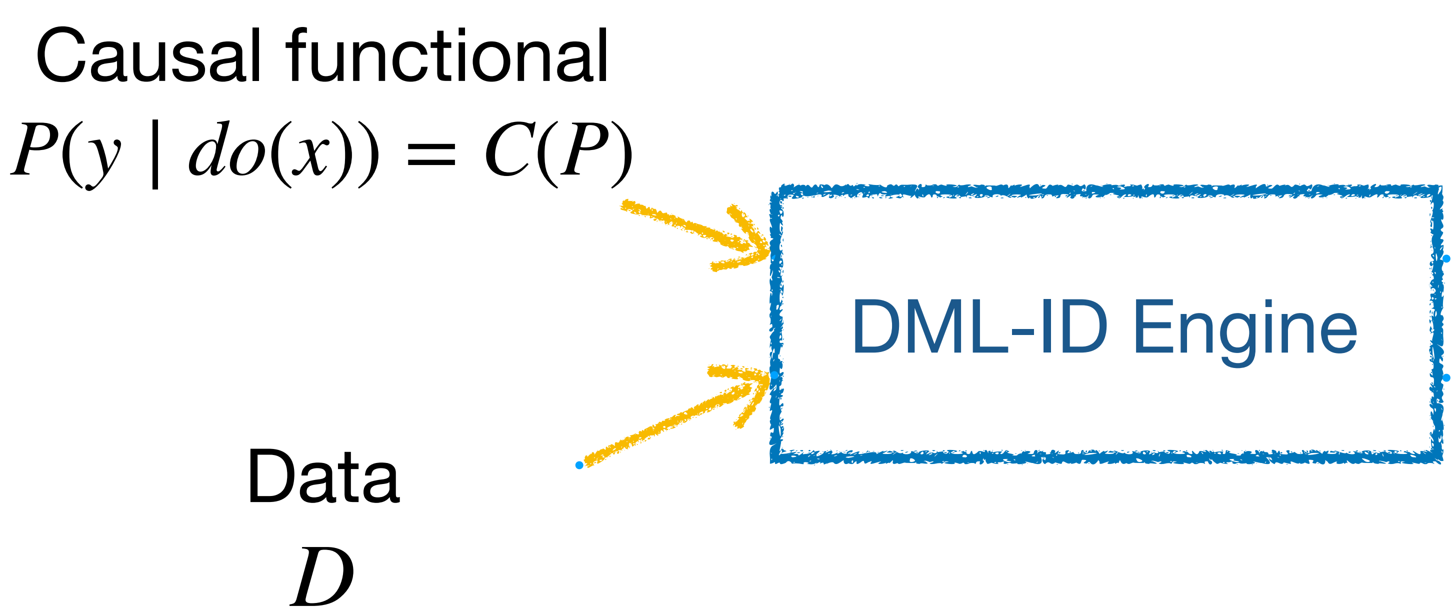
- **(Debiasedness; Left)** DML converges (i.e., the error decreases) faster even when nuisances converge slower rate ($N^{-1/4}$).

Empirical results



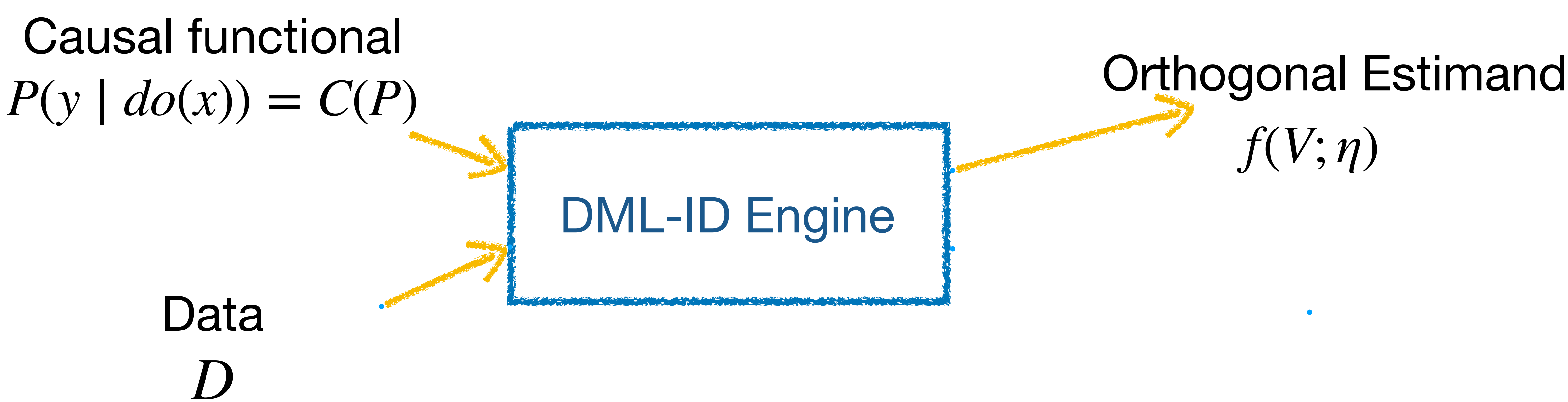
- (Debiasedness; Left) DML converges (i.e., the error decreases) faster even when nuisances converge slower rate ($N^{-1/4}$).
- (Doubly Robustness; (Center, Right)) DML converges even when models for either $P(x, y \mid r, w)$ (center) or $P(r \mid w)$ (right) is misspecified.

Conclusions



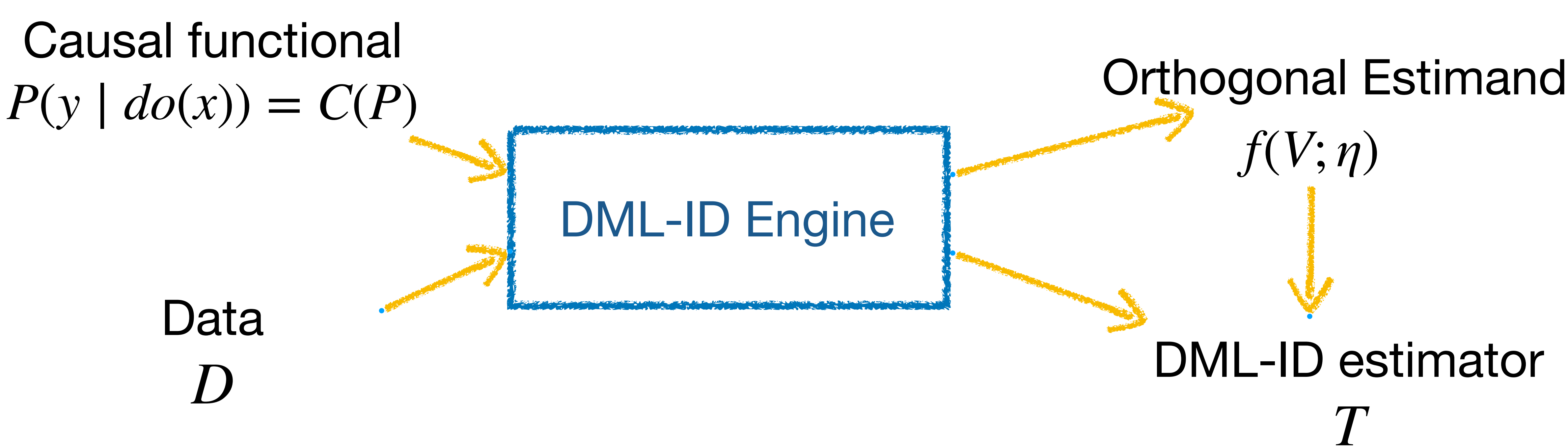
Conclusions

- We develop a systematic procedure for constructing a DML estimator for any identifiable causal effects.



Conclusions

- We develop a systematic procedure for constructing a DML estimator for any identifiable causal effects.
- A DML estimator enjoys *debiasedness* and *doubly robustness* against model misspecification and slow convergence rate.



Any Questions?